3.1 Conductors and insulators

The earliest experimenters with electricity observed that substances differed in their power to hold the "Electrick Vertue." Some materials could be easily electrified by friction and maintained in an electrified state; others, it seemed, could not be electrified that way, or did not hold the Vertue if they acquired it. Experimenters of the early eighteenth century compiled lists in which substances were classified as "electricks" or "nonelectricks." Around 1730, the important experiments of Stephen Gray in England showed that the Electrick Vertue could be conducted from one body to another by horizontal string, over distances of several hundred feet, provided that the string was itself supported from above by silk threads.[1] Once this distinction between conduction and nonconduction had been grasped, the electricians of the day found that even a nonelectrick could be highly electrified if it were supported on glass or suspended by silk threads. A spectacular conclusion of one of the popular electric exhibitions of the time was likely to be the electrification of a boy suspended by many silk threads from the rafters; his hair stood on end and sparks could be drawn from the tip of his nose.

After the work of Gray and his contemporaries, the elaborate lists of electricks and nonelectricks were seen to be, on the whole, a division of materials into electrical *insulators* and electrical *conductors*. This distinction is still one of the most striking and extreme contrasts that nature exhibits. Common good conductors like ordinary metals differ in their electrical conductivity from common insulators like glass and plastics by factors on the order of 10^{20}. To express it in a way the eighteenth-century experimenters like Gray or Benjamin Franklin would have understood, a metal globe on a metal post can lose its electrification in a millionth of a second; a metal globe on a glass post can hold its Vertue for many years. (To make good on the last assertion we would need to take some precautions beyond the capability of an eighteenth-century laboratory. Can you suggest some of them?)

The electrical difference between a good conductor and a good insulator is as vast as the mechanical difference between a liquid and a solid. That is not entirely accidental. Both properties depend on the *mobility* of atomic particles: in the electrical case, the mobility of the carriers of charge, electrons or ions; in the case of the mechanical properties, the mobility of the atoms or molecules that make up the structure of the material. To carry the analogy a bit further, we know of substances whose fluidity is intermediate between that of a solid and that of a liquid – substances such as tar or ice cream. Indeed some substances – glass is a good example – change gradually and continuously from a mobile

[1] The "pack-thread" Gray used for his string was doubtless a rather poor conductor compared with metal wire, but good enough for transferring charge in electrostatic experiments. Gray found, too, that fine copper wire was a conductor, but mostly he used the pack-thread for the longer distances.

liquid to a very permanent and rigid solid with a few hundred degrees' lowering of the temperature. In electrical conductivity, too, we find examples over the whole wide range from good conductor to good insulator, and some substances that can change conductivity over nearly as wide a range, depending on conditions such as their temperature. A fascinating and useful class of materials called semiconductors, which we shall meet in Chapter 4, have this property.

Whether we call a material solid or liquid sometimes depends on the time scale, and perhaps also on the scale of distances involved. Natural asphalt seems solid enough if you hold a chunk in your hand. Viewed geologically, it is a liquid, welling up from underground deposits and even forming lakes. We may expect that, for somewhat similar reasons, whether a material is to be regarded as an electrical insulator or a conductor will depend on the time scale of the phenomenon we are interested in.

3.2 Conductors in the electrostatic field

We shall look first at electrostatic systems involving conductors. That is, we shall be interested in the *stationary* state of charge and electric field that prevails after all redistributions of charge have taken place in the conductors. Any insulators present are assumed to be perfect insulators. As we have already mentioned, quite ordinary insulators come remarkably close to this idealization, so the systems we shall discuss are not too artificial. In fact, the air around us is an extremely good insulator. The systems we have in mind might be typified by some such example as this: bring in two charged metal spheres, insulated from one another and from everything else. Fix them in positions relatively near one another. What is the resulting electric field in the whole space surrounding and between the spheres, and how is the charge that is on each sphere distributed? We begin with a more general question: after the charges have become stationary, what can we say about the electric field inside conducting matter?

In the static situation there is no further motion of charge. You might be tempted to say that the electric field must then be zero within conducting material. You might argue that, if the field were *not* zero, the mobile charge carriers would experience a force and would be thereby set in motion, and thus we would not have a static situation after all. Such an argument overlooks the possibility of *other* forces that may be acting on the charge carriers, and that would have to be counterbalanced by an electric force to bring about a stationary state. To remind ourselves that it is physically possible to have other than electrical forces acting on the charge carriers, we need only think of gravity. A positive ion has weight; it experiences a steady force in a gravitational field, and so does an electron; also, the forces they experience are not equal. This is a rather absurd example. We know that gravitational forces are utterly negligible on an atomic scale.

There are other forces at work, however, which we may very loosely call "chemical." In a battery and in many, many other theaters of chemical reaction, including the living cell, charge carriers sometimes move *against* the general electric field; they do so because a reaction may thereby take place that yields more energy than it costs to buck the field. One hesitates to call these forces nonelectrical, knowing as we do that the structure of atoms and molecules and the forces between them can be explained in terms of Coulomb's law and quantum mechanics. Still, from the viewpoint of our *classical* theory of electricity, they must be treated as quite extraneous. Certainly they behave very differently from the inverse-square force upon which our theory is based. The general necessity for forces that are in this sense nonelectrical was already foreshadowed by our discovery in Chapter 2 that inverse-square forces alone cannot make a stable, static structure (see Earnshaw's theorem in Section 2.12).

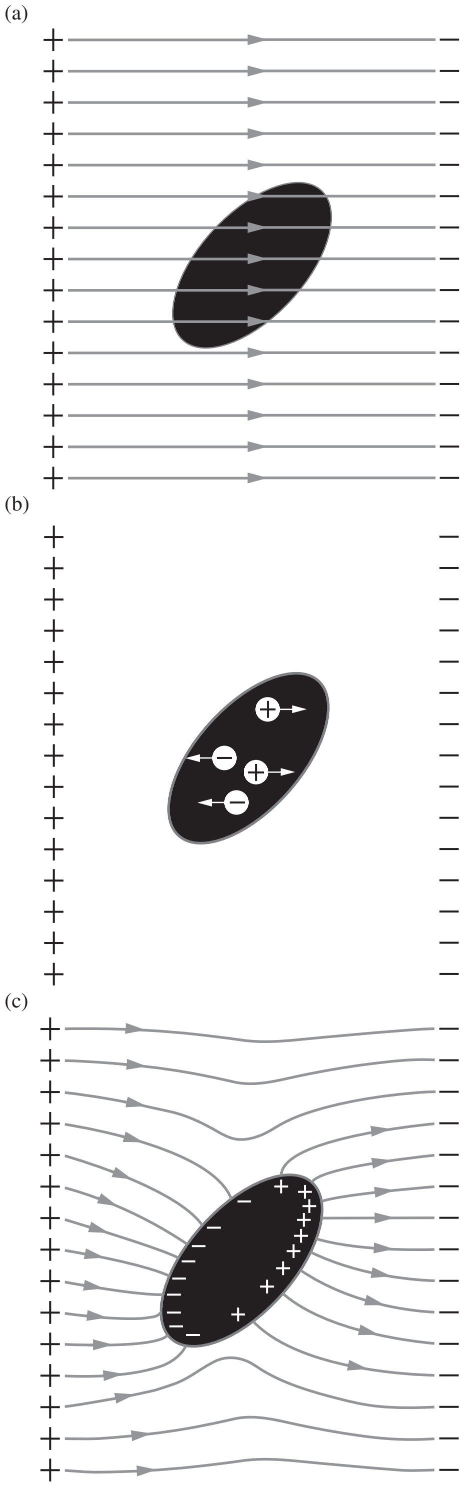
The point is simply this: we must be prepared to find, in some cases, unbalanced, non-Coulomb forces acting on charge carriers inside a conducting medium. When that happens, the electrostatic situation is attained when there *is* a finite electric field in the conductor that just offsets the influence of the other forces, whatever they may be.

Having issued this warning, however, we turn at once to the very familiar and important case in which there is no such force to worry about, the case of a homogeneous, isotropic conducting material. In the interior of such a conductor, in the static case, we can state confidently that the electric field must be zero.[2] If it weren't, charges would have to move. It follows that all regions inside the conductor, including all points just below its surface, must be at the same potential. Outside the conductor, the electric field is not zero. The surface of the conductor must be an equipotential surface of this field.

The vanishing of the electric field in the interior of a conductor implies that the volume charge density ρ also vanishes in the interior. This follows from Gauss's law, $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$. Since the field is identically zero inside the conductor, its divergence, and hence ρ, are also identically zero. Of course, as with the field, this holds only in an average sense. The charge density at the location of, say, a proton is most certainly not zero.

Imagine that we could change a material from insulator to conductor at will. (It's not impossible – glass becomes conducting when heated; any gas can be ionized by x-rays.) Figure 3.1(a) shows an uncharged nonconductor in the electric field produced by two fixed layers of charge.

[2] In speaking of the electric field inside matter, we mean an average field, averaged over a region large compared with the details of the atomic structure. We know, of course, that very strong fields exist in all matter, including the good conductors, if we search on a small scale near an atomic nucleus. The nuclear electric field does not contribute to the average field in matter, ordinarily, because it points in one direction on one side of a nucleus and in the opposite direction on the other side. Just how this average field ought to be defined, and how it could be measured, are questions we consider in Chapter 10.



The electric field is the same inside the body as outside. (A dense body such as glass would actually distort the field, an effect we will study in Chapter 10, but that is not important here.) Now, in one way or another, let mobile charges (or *ions*) be created, making the body a conductor. Positive ions are drawn in one direction by the field, negative ions in the opposite direction, as indicated in Fig. 3.1(b). They can go no farther than the surface of the conductor. Piling up there, they begin themselves to create an electric field inside the body which tends to *cancel* the original field. And in fact the movement goes on until that original field is *precisely* canceled. The final distribution of charge at the surface, shown in Fig. 3.1(c), is such that its field and the field of the fixed external sources combine to give *zero* electric field in the interior of the conductor. Because this "automatically" happens in every conductor, it is really only the surface of a conductor that we need to consider when we are concerned with the external fields.

With this in mind, let us see what can be said about a system of conductors, variously charged, in otherwise empty space. In Fig. 3.2 we see some objects. Think of them, if you like, as solid pieces of metal. They are prevented from moving by invisible insulators – perhaps by Stephen Gray's silk threads. The total charge of each object, by which we mean the net excess of positive over negative charge, is fixed because there is no way for charge to leak on or off. We denote it by Q_k, for the kth conductor. Each object can also be characterized by a particular value ϕ_k of the electric potential function ϕ. We say that conductor 2 is "at the potential ϕ_2." With a system like the one shown, where no physical objects stretch out to infinity, it is usually convenient to assign the potential zero to points infinitely far away. In that case ϕ_2 is the work per unit charge required to bring an infinitesimal test charge in from infinity and put it anywhere on conductor 2. (Note, by the way, that this is just the kind of system in which the test charge needs to be kept small, a point raised in Section 1.7.)

Because the surface of a conductor in Fig. 3.2 is necessarily a surface of constant potential, the electric field, which is $-\text{grad}\,\phi$, must be *perpendicular* to the surface at every point on the surface. Proceeding from the interior of the conductor outward, we find at the surface an abrupt change in the electric field; $\mathbf{E}$ is not zero outside the surface, and it is zero inside. The discontinuity in $\mathbf{E}$ is accounted for by the presence of a surface charge, of density σ, which we can relate directly to $\mathbf{E}$ by Gauss's law. We can use a flat box enclosing a patch of surface (Fig. 3.3), similar to the cylinder we used when considering the infinite

Figure 3.1.
The object in (a) is a neutral nonconductor. The charges in it, both positive and negative, are immobile. In (b) the charges have been released and begin to move. They will move until the final condition, shown in (c), is attained.

flat sheet in Section 1.13. However, here there is *no* flux through the "bottom" of the box, which lies inside the conductor, so we conclude that $E_n = \sigma/\epsilon_0$ (instead of the $\sigma/2\epsilon_0$ we found in Eq. (1.40)), where E_n is the component of electric field normal to the surface. As we have already seen, there *is* no other component in this case, the field being always perpendicular to the surface. The surface charge must account for the whole charge Q_k. That is, the surface integral of σ over the whole conductor must equal Q_k. In summary, we can make the following statements about *any* such system of conductors, whatever their shape and arrangement:

(1) $\mathbf{E} = 0$ inside the material of a conductor;

(2) $\rho = 0$ inside the material of a conductor;

(3) $\phi = \phi_k$ at all points inside the material and on the surface of the kth conductor;

(4) At any point just outside the conductor, $\mathbf{E}$ is perpendicular to the surface, and $E = \sigma/\epsilon_0$, where σ is the local density of surface charge;

(5) $Q_k = \int_{S_k} \sigma\, da = \epsilon_0 \int_{S_k} \mathbf{E} \cdot d\mathbf{a}$.

$\mathbf{E}$ is the total field arising from *all* the charges in the system, near and far, of which the surface charge is only a part. The surface charge on a conductor is obliged to "readjust itself" until relation (4) is fulfilled. That the conductor presents a special case, in contrast to other surface charge distributions, is brought out by the comparison in Fig. 3.4.

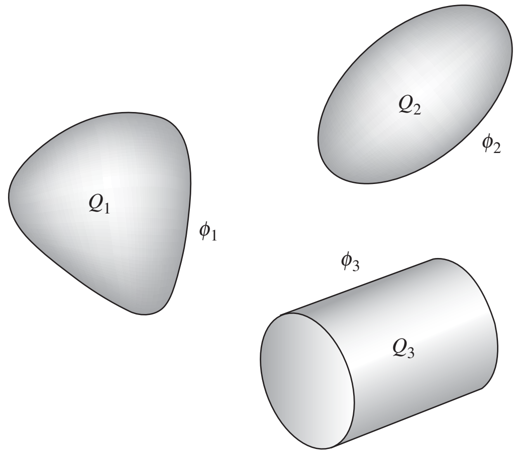


Figure 3.2.
A system of three conductors: Q_1 is the charge on conductor 1, ϕ_1 is its potential, etc.

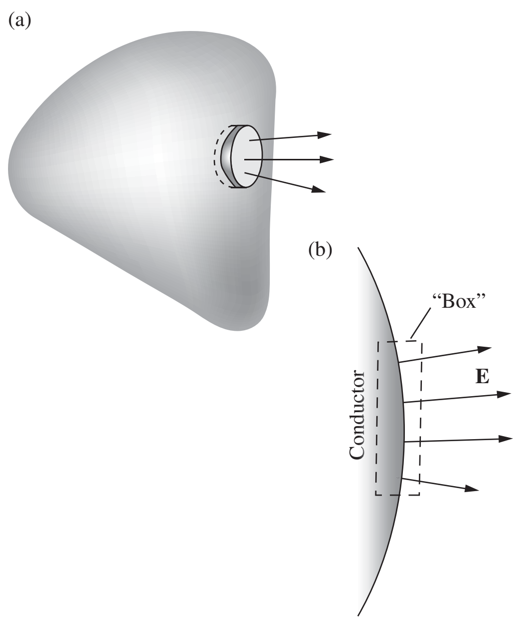


Figure 3.3.
(a) Gauss's law relates the electric field strength at the surface of a conductor to the density of surface charge; $E = \sigma/\epsilon_0$. (b) Cross section through surface of conductor and box.

Example (A spherically symmetric field) A point charge q is located at an arbitrary position inside a neutral conducting spherical shell. Explain why the electric field outside the shell is the same as the spherically symmetric field due to a charge q located at the *center* of the shell (with the shell removed, although the point is that this doesn't matter).

Solution The spherical shell has an inner surface and an outer surface. Between these surfaces (inside the material of the conductor) we know that the electric field is zero. So if we draw a Gaussian surface that lies entirely inside the material, signified by the dashed line in Fig. 3.5, there is zero flux through it, so it must enclose zero charge. The charge on the inner surface of the shell is therefore $-q$. This leaves $+q$ for the outer surface. The charge $-q$ on the inner surface won't be uniformly distributed unless the point charge q is located at the center, but that doesn't concern us.

The only question is how the $+q$ charge is distributed over the outer surface. Imagine that we have removed this $+q$ charge, so that we have only the point charge q and the inner-surface charge $-q$. The combination of these charges produces zero field in the material of the conductor. It also produces zero field outside the conductor. This is true because field lines must have at least one end on a charge (the other end may be at infinity); they can't form closed loops because the electric field has zero curl. However, in the present setup, external field lines have no possibility of touching any of the charges on the inside, because the lines can't pass through the material of the conductor to reach them, since the field is zero there. Therefore there can be no field lines outside the conductor.

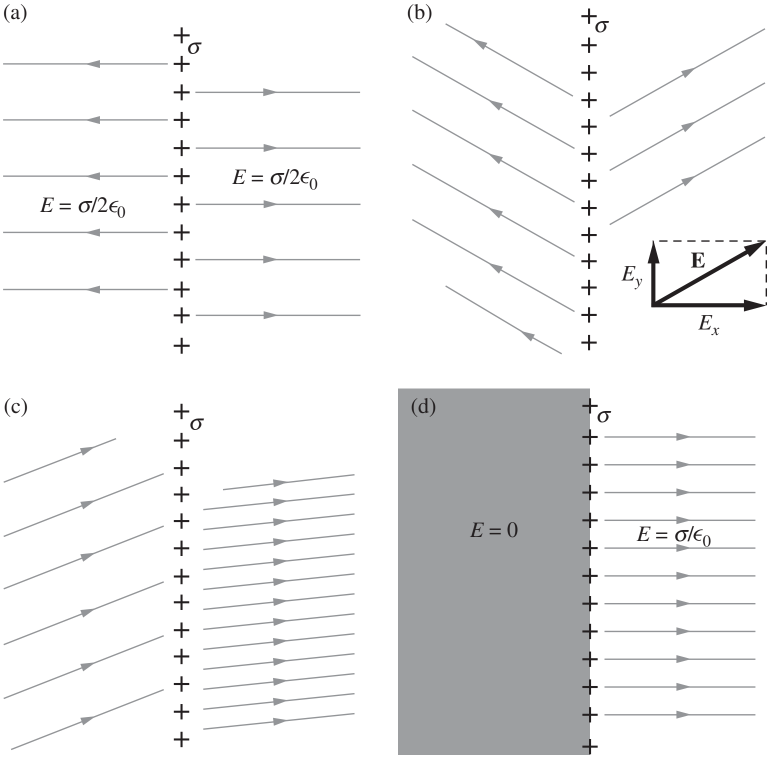


Figure 3.4.
(a) An isolated sheet of surface charge with nothing else in the system. This was treated in Fig. 1.26. The field was determined as $\sigma/2\epsilon_0$ on each side of the sheet by the assumption of symmetry. (b) If there are other charges in the system, we can say only that the change in E_x at the surface must be σ/ϵ_0, with zero change in E_y. Many fields other than the field of (a) above could have this property. Two such are shown in (b) and (c). (d) If we know that the medium on one side of the surface is a conductor, we know that on the other side $\mathbf{E}$ must be perpendicular to the surface, with magnitude $E = \sigma/\epsilon_0$. $\mathbf{E}$ could not have a component parallel to the surface without causing charge to move.

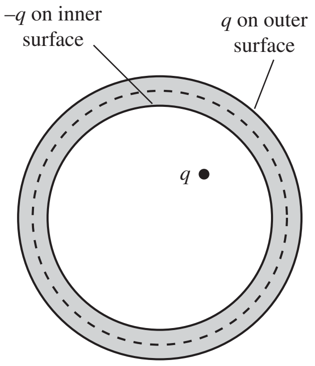


Figure 3.5.
A Gaussian surface (dashed line) inside the material of a conducting spherical shell.

If we gradually add back on the outer-surface charge $+q$, it will distribute itself in a spherically symmetric manner because it feels no field from the other charges. Furthermore, due to this spherical symmetry, the outer-surface charge will produce no field at the other charges (because a uniform shell produces zero field in its interior), so we don't have to worry about any shifting of these charges.

Since the combination of the point charge and the inner-surface charge produces no field outside the shell, the external field is due only to the spherically symmetric outer-surface charge. By Gauss's law, the external field is therefore radial (with respect to the center of the shell and *not* the point charge q) and has magnitude $q/4\pi\epsilon_0 r^2$. Note that the shape of the inner surface was irrelevant in the above reasoning. If we have the setup shown in Fig. 3.6, the external field is still spherically symmetric with magnitude $q/4\pi\epsilon_0 r^2$.

More generally, if the neutral conducting shell takes an odd nonspherical shape, we can't say that the external field is spherically symmetric. But we *can* say that the external field, whatever it may be, is *independent of the location* of the point charge q inside. Whatever the location, the external field equals the field in a system where the point charge q is absent and where we instead dump a total charge q on the shell (which will distribute itself in a particular manner).[3]

[3] There is a slight subtlety that arises in this case, namely the effect of the outer-surface charge on the inner-surface charge. It turns out that, as with the sphere, there is no effect. We'll see why in Section 3.3.

Figure 3.7 shows the field and charge distribution for a simple system like the one mentioned at the beginning of this section. There are two conducting spheres, a sphere of unit radius carrying a total charge of +1 unit, the other a somewhat larger sphere with total charge zero. Observe that the surface charge density is not uniform over either of the conductors. The sphere on the right, with total charge zero, has a negative surface charge density in the region that faces the other sphere, and a positive surface charge on the rearward portion of its surface. The dashed curves in Fig. 3.7 indicate the equipotential surfaces or, rather, their intersection with the plane of the figure. If we were to go a long way out, we would find the equipotential surfaces becoming nearly spherical and the field lines nearly radial, and the field would begin to look very much like that of a point charge of magnitude 1 and positive, which is the net charge on the entire system.

Figure 3.7 illustrates, at least qualitatively, all the features we anticipated, but we have an additional reason for showing it. Simple as the system is, the exact mathematical solution for this case cannot be obtained

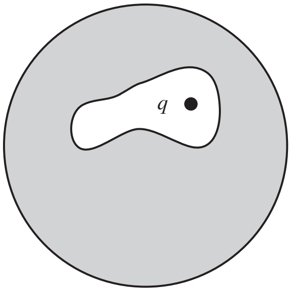


Figure 3.6.
The external field is radial even if the cavity takes an odd shape.

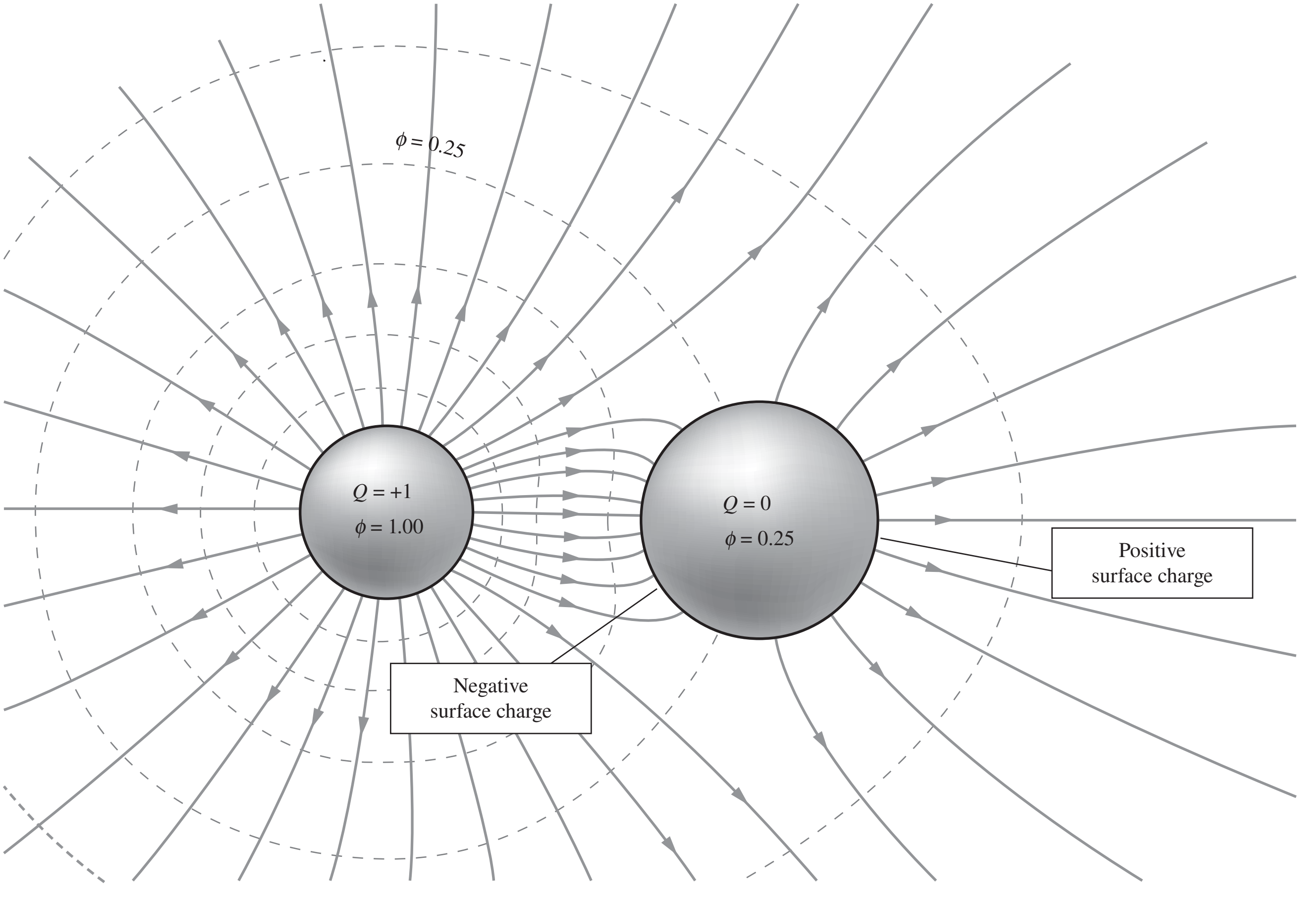


Figure 3.7.
The electric field around two spherical conductors, one with total charge +1, and one with total charge zero. Dashed curves are intersections of equipotential surfaces with the plane of the figure. Zero potential is at infinity.

in a straightforward way. Our Fig. 3.7 was constructed from an approximate solution. In fact, the number of three-dimensional geometrical arrangements of conductors that permit a mathematical solution in closed form is lamentably small. One does not learn much physics by concentrating on the solution of the few neatly soluble examples. Let us instead try to understand the general nature of the mathematical problem such a system presents.

3.3 The general electrostatic problem and the uniqueness theorem

We can state the problem in terms of the potential function ϕ, for if ϕ can be found, we can at once get $\mathbf{E}$ from it. Everywhere outside the conductors, ϕ has to satisfy the partial differential equation we met in Section 2.12, Laplace's equation: $\nabla^2\phi = 0$. Written out in Cartesian coordinates, Laplace's equation reads

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{3.1}$$

The problem is to find a function that satisfies Eq. (3.1) and also meets the specified conditions on the conducting surfaces. These conditions might have been set in various ways. It might be that the potential of each conductor ϕ_k is fixed or known. (In a real system the potentials may be fixed by permanent connections to batteries or other constant-potential "power supplies.") Then our solution $\phi(x, y, z)$ has to assume the correct value at all points on each of the surfaces. These surfaces in their totality *bound* the region in which ϕ is defined, if we include a large surface "at infinity," where we require ϕ to approach zero. Sometimes the region of interest is totally enclosed by a conducting surface; then we can assign this conductor a potential and ignore anything outside it. In either case, we have a typical *boundary-value problem,* in which the value the function has to assume on the boundary is specified for the entire boundary.

One might, instead, have specified the total charge on each conductor, Q_k. (We could not specify arbitrarily all charges and potentials; that would overdetermine the problem.) With the charges specified, we have in effect fixed the value of the surface integral of $\nabla\phi$ over the surface of each conductor (using fact (5) from Section 3.2, along with $\mathbf{E} = -\nabla\phi$). This gives the mathematical problem a slightly different aspect. Or one can "mix" the two kinds of boundary conditions.

A general question of some interest is this: with the boundary conditions given in some way, does the problem have no solution, one solution, or more than one solution? We shall not try to answer this question in all the forms it can take, but one important case will show how such questions can be dealt with and will give us a useful result. Suppose the potential of each conductor, ϕ_k, has been specified, together with the requirement that ϕ approach zero at infinite distance, or on a conductor that encloses the system. We shall prove that this boundary-value problem has no more than one solution. It seems obvious, as a matter of physics, that it has *a* solution, for if we should actually arrange the conductors in the prescribed manner, connecting them by infinitesimal wires to the proper potentials, the system would have to settle down in *some* state. However, it is quite a different matter to prove mathematically that a solution always exists, and we shall not attempt it. Instead, we shall prove the following theorem.

Theorem 3.1 *(Uniqueness theorem) Assuming that there* is *a solution $\phi(x, y, z)$ for a given set of conductors with potentials ϕ_k, this solution must be unique.*

Proof The argument, which is typical of proofs of this sort, runs as follows. Assume there is another function $\psi(x, y, z)$ that is also a solution meeting the same boundary conditions. Now Laplace's equation is *linear.* That is, if ϕ and ψ satisfy Eq. (3.1), then so does $\phi + \psi$ or any linear combination such as $c_1\phi + c_2\psi$, where c_1 and c_2 are constants. In particular, the difference between our two solutions, $\phi - \psi$, must satisfy Eq. (3.1). Call this function W:

$$W(x, y, z) \equiv \phi(x, y, z) - \psi(x, y, z). \tag{3.2}$$

Of course, W does *not* satisfy the boundary conditions. In fact, at the surface of every conductor W is zero, because ϕ and ψ take on the same value, ϕ_k, at the surface of a conductor k. Thus W is a solution of *another* electrostatic problem, one with the same conductors but with all conductors held at zero potential.

We can now assert that if W is zero on all the conductors, then W must be zero at all points in space. For if it is not, it must have either a maximum or a minimum somewhere – remember that W is zero at infinity as well as on all the conducting boundaries. If W has an extremum at some point P, consider a sphere centered on that point. As we saw in Section 2.12, the average over a sphere of a function that satisfies Laplace's equation is equal to its value at the center. This could not be true if the center is a maximum or minimum. Thus W cannot have a maximum or minimum;[4] it must therefore be zero everywhere. It follows that $\psi = \phi$ everywhere, that is, there can be only *one* solution of Eq. (3.1) that satisfies the prescribed boundary conditions. □

In proving this theorem, we assumed that ϕ and ψ satisfied Laplace's equation. That is, we assumed that the region outside the conductors was empty of charge. However, the uniqueness theorem actually holds even if

[4] If you want to demonstrate this without invoking the "average over a sphere" fact, you can use the related reasoning involving Gauss's law: if the potential at P is a maximum (or minimum), then $\mathbf{E}$ must point outward (or inward) everywhere around P. This implies a net flux through a small sphere surrounding P, contradicting the fact that there are no charges enclosed.

there are charges present, provided that these charges are fixed in place. These charges could come in the form of point charges or a continuous distribution. The proof for this more general case is essentially the same. In the above reasoning, you will note that we never used the fact that ϕ and ψ satisfied Laplace's equation, but rather only that their *difference* W did. So if we instead start with the more general Poisson's equations, $\nabla^2\phi = -\rho/\epsilon_0$ and $\nabla^2\psi = -\rho/\epsilon_0$, where the *same* ρ appears in both of these equations, then we can take their difference to obtain $\nabla^2 W = 0$. That is, W satisfies Laplace's equation. The proof therefore proceeds exactly as above, and we again obtain $\phi = \psi$.

As a quick corollary to the uniqueness theorem, we can demonstrate a remarkable fact as follows.

Corollary 3.2 *In the space inside a hollow conductor of any shape whatsoever, if that space itself is empty of charge, the electric field is zero.*

Proof The potential function inside the conductor, $\phi(x, y, z)$, must satisfy Laplace's equation. The entire boundary of this region, namely the conductor, is an equipotential, so we have $\phi = \phi_0$, a constant everywhere on the boundary. One solution is obviously $\phi = \phi_0$ throughout the volume. But there can be only one solution, according to the above uniqueness theorem, so this is it. And then "$\phi = \text{constant}$" implies $\mathbf{E} = 0$, because $\mathbf{E} = -\nabla\phi$. □

This corollary is true whatever the field may be outside the conductor. We are already familiar with the fact that the field is zero inside an isolated uniform spherical shell of charge, just as the gravitational field inside the shell of a hollow spherical mass is zero. The corollary we just proved is, in a way, more surprising. Consider the closed metal box shown partly cut away in Fig. 3.8. There are charges in the neighborhood of the box, and the external field is approximately as depicted. There is a highly nonuniform distribution of charge over the surface of the box. Now the field everywhere in space, *including the interior of the box,* is the sum of the field of this charge distribution and the fields of the external sources. It seems hardly credible that the surface charge has so cleverly arranged itself on the box that its field precisely *cancels* the field of the external sources at every point inside the box. Yet this must indeed be what has happened, in view of the above proof.

As surprising as this may seem for a hollow conductor, it is really no more surprising than the fact that the charges on the surface of a *solid* conductor arrange themselves so that the electric field is zero inside the material of the conductor (which we know is the case, otherwise charges in the interior would move). These two setups are related because the interior of the solid conductor is neutral (since $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, and $\mathbf{E}$ is identically zero). So if we remove this neutral material from the solid conductor (a process that can't change the electric field anywhere,

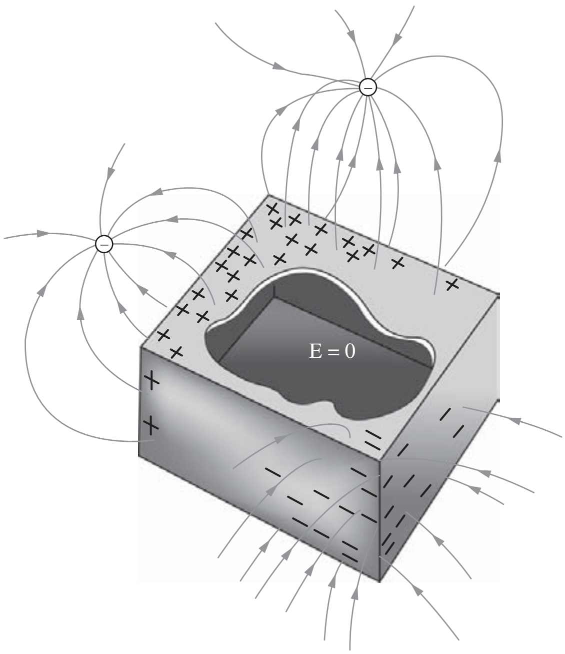


Figure 3.8.
The field is zero everywhere inside a closed conducting box.

because we aren't moving any particles with net charge), then we end up with a hollow conductor with zero field inside.

The corollary is also consistent with what we know about field lines. If there were field lines inside the shell, they would have to start at one point on the shell and end at another (there can't be any closed loops because curl $\mathbf{E} = 0$). But this would imply a nonzero potential difference between these two points on the shell, contradicting the fact that all points on the shell have the same potential. Therefore there can be no field lines inside the shell.

The absence of electric field inside a conducting enclosure is useful, as well as theoretically interesting. It is the basis for electrical shielding. For most practical purposes the enclosure does not need to be completely tight. If the walls are perforated with small holes, or made of metallic screen, the field inside will be extremely weak except in the immediate vicinity of a hole. A metal pipe with open ends, if it is a few diameters long, very effectively shields the space inside that is not close to either end. We are considering only static fields of course, but for slowly varying electric fields these remarks still hold. (A rapidly varying field can become a wave that travels through the pipe. *Rapidly* means here "in less time than light takes to travel a pipe diameter.")

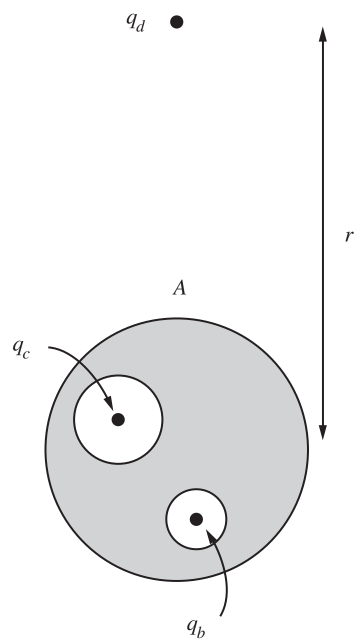


Figure 3.9.
Point charges are located at the centers of spherical cavities inside a neutral spherical conductor. Another point charge is located far away.

Example (Charges in cavities) A spherical conductor A contains two spherical cavities. The total charge on the conductor itself is zero. However, there is a point charge q_b at the center of one cavity and q_c at the center of the other, as shown in Fig. 3.9. A considerable distance r away is another charge q_d. What force acts on each of the four objects, A, q_b, q_c, q_d? Which answers, if any, are only approximate, and depend on r being relatively large?

Solution The short answer is that the forces on q_b and q_c are exactly zero, and the forces on A and q_d are exactly equal and opposite, with a magnitude approximately equal to $q_d(q_b + q_c)/4\pi\epsilon_0 r^2$. The reasoning is as follows.

Let's look at q_b first; the reasoning for q_c is the same. If the charge q_b *weren't* present in the lower cavity, then the field inside this cavity would be zero, due to the uniqueness theorem, as discussed above. This fact is independent of whatever is going on with q_c and q_d. If we now reintroduce q_b at the center of the cavity, this induces a total charge $-q_b$ on the surface of the cavity (as we saw in the example in Section 3.2). This charge is uniformly distributed over the surface because q_b is located at the center. This charge therefore doesn't change the fact that the field is zero at the center of the cavity. The force on q_b is therefore zero. The same reasoning applies to q_c. Note that the force on q_b would *not* be zero if it were located off-center in the cavity.

Now let's look at the conductor A. Since the total charge on A is zero, a charge of $q_b + q_c$ must be distributed over its outside surface, to balance the $-q_b$ and $-q_c$ charges on the surfaces of the cavities. If q_d were absent, the field outside A would be the symmetrical radial field, $E = (q_b + q_c)/4\pi\epsilon_0 r^2$, with the charge $q_b + q_c$ uniformly distributed over the outside surface. The distribution would indeed be uniform because the field inside the material of the conductor is zero, and because we are assuming that there is no charge external to the conductor. The setup is therefore spherically symmetric, as far as the outside surface of the conductor is concerned. (Any effect of the interior charges on the outside surface charge can be felt only through the field. And since the field is zero just inside the outside surface, there is therefore no effect.)

If we now reintroduce the charge q_d, its influence will slightly alter the distribution of charge on the outside surface of A, but without affecting the total amount. If q_d is positive, then negative charge will be drawn toward the near side of A, or equivalently positive charge will be pushed to the far side. Hence for large r, the force on q_d will be approximately equal to $q_d(q_b + q_c)/4\pi\epsilon_0 r^2$, but it will be slightly more attractive than this; you can check that this is true for either sign of $q_d(q_b + q_c)$. The force on A must be exactly equal and opposite to the force on q_d.

The *exact* value of the force on q_d is the sum of the force just given, $q_d(q_b + q_c)/4\pi\epsilon_0 r^2$, and the force that would act on q_d if the total charge *on and within* A were zero (it is $q_b + q_c$ here). This latter force (which is always attractive) can be determined by applying the "image charge" technique that we will learn about in the following section; see Problem 3.13.

3.4 Image charges

About the simplest system in which the mobility of the charges in the conductor makes itself evident is the point charge near a conducting plane. Suppose the xy plane is the surface of a conductor extending out to infinity. Let's assign this plane the potential zero. Now bring in a positive charge Q and locate it h above the plane on the z axis, as in Fig. 3.10(a). What sort of field and charge distribution can we expect? We expect the positive charge Q to attract negative charge, but we hardly expect the negative charge to pile up in an infinitely dense concentration at the foot of the perpendicular from Q. (Why not?) Also, we remember that the electric field is always perpendicular to the surface of a conductor, at the conductor's surface. Very near the point charge Q, on the other hand, the presence of the conducting plane can make little difference; the field lines must *start out* from Q as if they were leaving a point charge radially. So we might expect something qualitatively like Fig. 3.10(b), with some of the details still a bit uncertain. Of course the whole thing is bound to be quite symmetrical about the z axis.

But how do we really solve the problem? The answer is, by a trick, but a trick that is both instructive and frequently useful. We find an easily soluble problem whose solution, or a piece of it, can be made to fit the problem at hand. Here the easy problem is that of two equal and opposite point charges, Q and $-Q$. On the plane that bisects the line joining the two charges, the plane indicated in cross section by the line AA in Fig. 3.10(c), the electric field is everywhere perpendicular to the plane. If we make the distance of Q from the plane agree with the distance h in our original problem, the upper half of the field in Fig. 3.10(c) meets all our requirements: the field is perpendicular to the plane of the conductor, and in the neighborhood of Q it approaches the field of a point charge.

The boundary conditions here are not quite those that figured in our uniqueness theorem in Section 3.3. The potential of the conductor is fixed, but we have in the system a point charge at which the potential approaches infinity. We can regard the point charge as the limiting case of a small, spherical conductor on which the total charge Q is fixed. For this mixed boundary condition – potentials given on some surfaces, total charge on others – a uniqueness theorem also holds. If our "borrowed" solution fits the boundary conditions, it must be *the* solution.

Figure 3.11 shows the final solution for the field above the plane, with the density of the surface charge suggested. We can calculate the field strength and direction at any point by going back to the two-charge problem, Fig. 3.10(c), and using Coulomb's law. Consider a point on the surface, a distance r from the origin. The square of its distance from Q is $r^2 + h^2$, and the z component of the field of Q, at this point, is $-Q\cos\theta/4\pi\epsilon_0(r^2+h^2)$. The "image charge," $-Q$, below the plane contributes an equal z component. Thus the electric field here is given by

$$E_z = \frac{-2Q}{4\pi\epsilon_0(r^2+h^2)}\cos\theta = \frac{-2Q}{4\pi\epsilon_0(r^2+h^2)}\cdot\frac{h}{(r^2+h^2)^{1/2}}$$
$$= \frac{-Qh}{2\pi\epsilon_0(r^2+h^2)^{3/2}}. \quad (3.3)$$

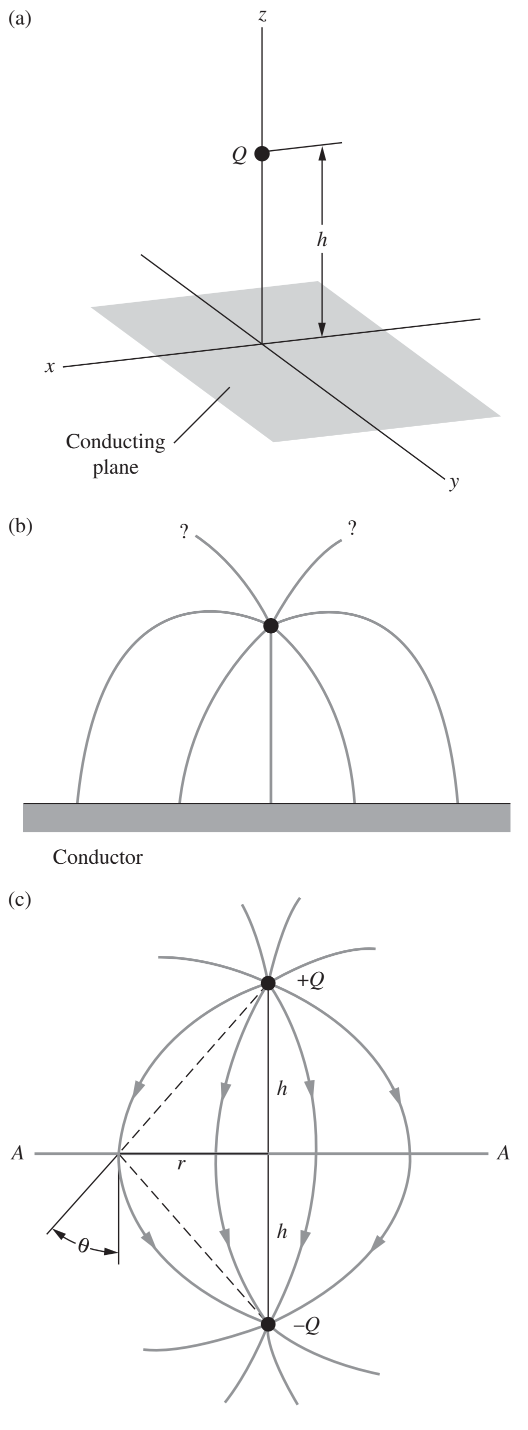


Figure 3.10.
(a) A point charge Q above an infinite plane conductor. (b) The field must look something like this. (c) The field of a pair of opposite charges.

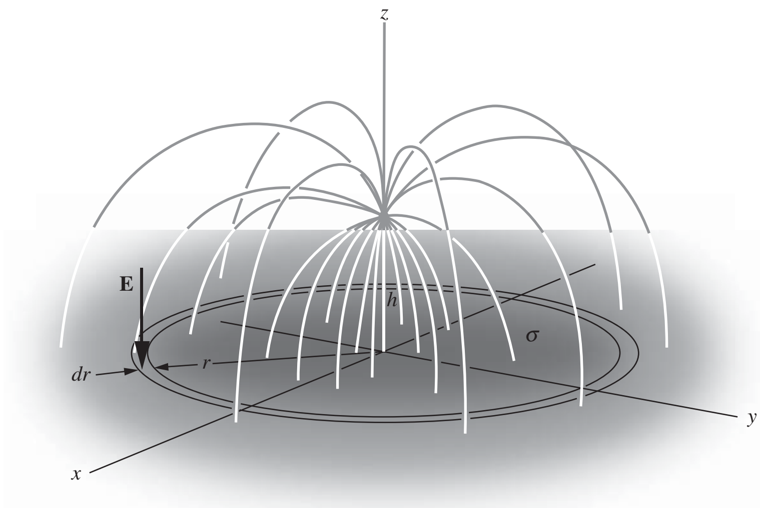


Figure 3.11.
Some field lines for the charge above the plane. The field strength at the surface, given by Eq. (3.3), determines the surface charge density σ.

Returning to the actual setup with the conducting plane, we know that in terms of the surface charge density σ, the electric field just above the plane is $E_z = \sigma/\epsilon_0$. There is no factor of 2 in the denominator here, because when using Gauss's law with a small pillbox, there is zero field below the conducting plane, so there is zero flux out the bottom of the box. The field is indeed zero below the plane because we can consider the conducting plane to be the top of a very large conducting sphere, and we know that the field inside a conductor is zero. Using $E_z = \sigma/\epsilon_0$, the density σ is given by

$$\sigma = \epsilon_0 E_z = \frac{-Qh}{2\pi(r^2 + h^2)^{3/2}}. \tag{3.4}$$

Let us calculate the total amount of charge on the surface by integrating over the distribution:

$$\int_0^\infty \sigma \cdot 2\pi r\, dr = -Qh \int_0^\infty \frac{r\, dr}{(r^2 + h^2)^{3/2}} = \frac{Qh}{(r^2 + h^2)^{1/2}}\Bigg|_0^\infty = -Q. \tag{3.5}$$

This result was to be expected. It means that all the flux leaving the charge Q ends on the conducting plane.

There is one puzzling point. We never said what the charge on the conducting plane was, but what if we had chosen it to be zero before the charge Q was put in place above it? (You might have just assumed this was the case anyway.) How can the conductor now exhibit a net charge $-Q$? The answer is that a compensating positive charge, $+Q$ in amount, must be distributed over the whole plane. The combination of the given point charge Q and the surface density σ in Eq. (3.4) produces the E_z

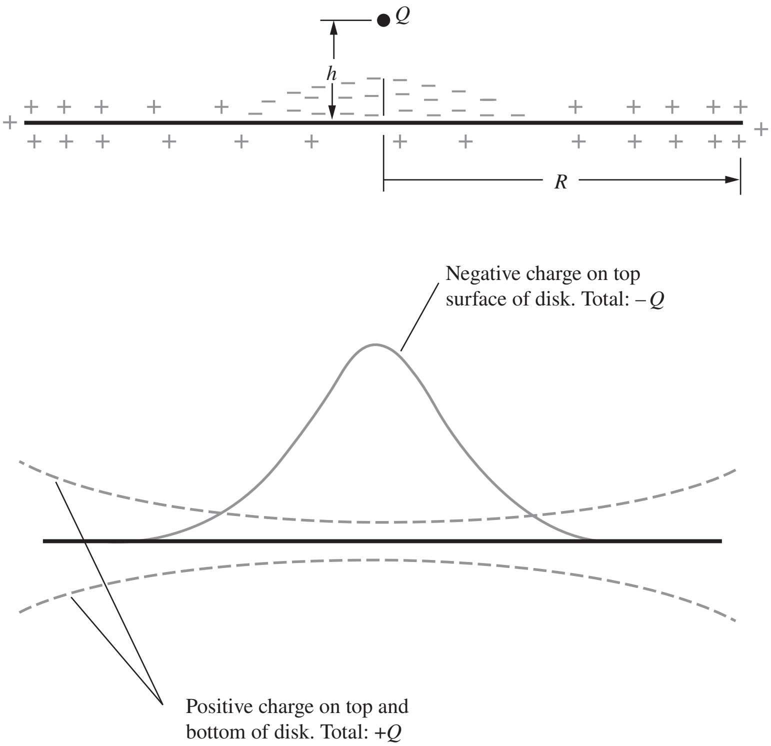


Figure 3.12.
The distribution of charge on a conducting disk with total charge zero, in the presence of a positive point charge Q at height h above the center of the disk. The actual surface charge density at any point is of course the algebraic sum of the positive and negative densities shown.

field in Eq. (3.3), but nothing precludes us from superposing additional charge on the conducting plane which will produce an additional field.

To see what is going on here, imagine that the conducting plane is actually a metal disk, not infinite but finite and with a radius $R \gg h$. If a charge $+Q$ were to be spread uniformly over this disk, on *both* sides (so $Q/2$ is on each side), the resulting surface density on each side would be $Q/2\pi R^2$, which would cause an electric field of strength $Q/2\pi\epsilon_0 R^2$ normal to the plane of the disk. Since our disk is a conductor, on which charge can move, the charge density and the resulting field strength will be even *less* than $Q/2\pi\epsilon_0 R^2$ near the center of the disk because of the tendency of the charge to spread out toward the rim. In any case the field of this distribution is smaller in order of magnitude by a factor h^2/R^2 than the field described by Eq. (3.3), because the latter field behaves like $1/h^2$ in the vicinity of $r = 0$. As long as $R \gg h$ we were justified in ignoring the former field, and of course it vanishes completely for an unbounded conducting plane with $R = \infty$.

Figure 3.12 shows in separate plots the surface charge density σ, given by Eq. (3.4), and the distribution of the compensating charge Q on the upper and lower surfaces of the disk. Here we have made R not very much larger than h, in order to show both distributions clearly in the same diagram. Note that the compensating positive charge has arranged

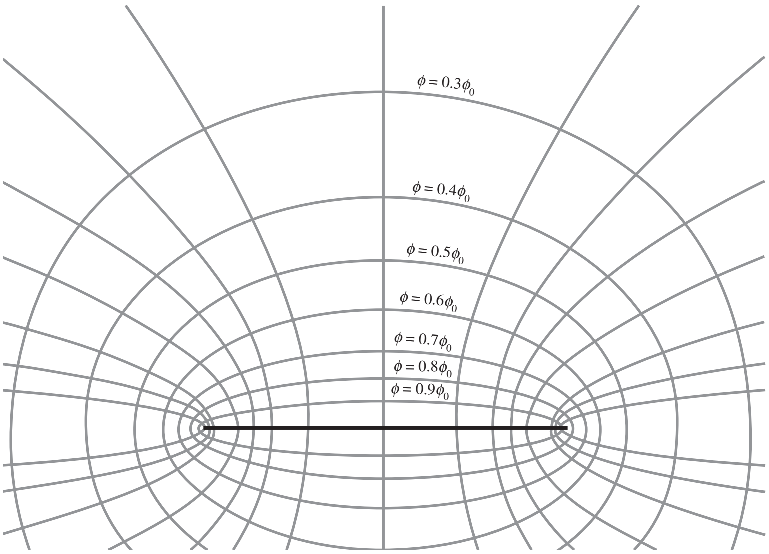


Figure 3.13.
Equipotentials and field lines for a charged conducting disk.

itself in exactly the same way on the top and bottom surfaces of the disk, as if it were utterly ignoring the pile of negative charge in the middle of the upper surface! Indeed, it is free to do so, for the field of that negative charge distribution *plus* that of the point charge Q that induced it has horizontal component zero at the surface of the disk, and hence has no influence whatsoever on the distribution of the compensating positive charge.

The isolated conducting disk mentioned above belongs to another class of soluble problems, a class that includes any isolated conductor in the shape of a spheroid, an ellipsoid of revolution. Without going into the mathematics[5] we show in Fig. 3.13 some electric field lines and equipotential surfaces around the conducting disk. The field lines are hyperbolas. The equipotentials are oblate ellipsoids of revolution enclosing the disk. The potential ϕ of the disk itself, relative to infinity, turns out to be

$$\phi_0 = \frac{(\pi/2)Q}{4\pi\epsilon_0 a}, \tag{3.6}$$

where Q is the total charge of the disk and a is its radius. (Written this way, we see that ϕ_0 is larger than the potential of a sphere of charge Q and radius a, by a factor $\pi/2$.) Compare this picture with Fig. 2.12, the field of a *uniformly* charged *non*conducting disk. In that case the electric field at the surface was not normal to the surface; it had a radial component outward. If you could make that disk in Fig. 2.12 a conductor, the charge would flow outward until the field in Fig. 3.13 was established.

[5] Mathematically speaking, this class of problems is soluble because a spheroidal coordinate system happens to be one of those systems in which Laplace's equation takes on a particularly simple form.

According to the mathematical solution on which Fig. 3.13 is based, the charge density at the center of the disk would then be just half as great as it was at the center of the uniformly charged disk. This fact also follows as a corollary to Problem 3.4.

Figure 3.13 shows us the field not only of the conducting disk, but also of any isolated oblate spheroidal conductor. To see that, choose one of the equipotential surfaces of revolution – say the one whose trace in the diagram is the ellipse marked $\phi = 0.6\,\phi_0$. Imagine that we could plate this spheroid with copper and deposit charge Q on it. Then the field shown outside it already satisfies the boundary conditions: electric field normal to surface; total flux Q/ϵ_0. It is *a* solution, and in view of the uniqueness theorem it must be *the* solution for an isolated charged conductor of that particular shape. All we need to do is erase the field lines *inside* the conductor. We can also imagine copperplating two of the spheroidal surfaces, putting charge Q on the inner surface, $-Q$ on the outer. The section of Fig. 3.13 between these two equipotentials shows us the field between two such concentric spheroidal conductors. The field is zero elsewhere.

This suggests a general strategy. Given the solution for any electrostatic problem with the equipotentials located, we can extract from it the solution for any other system made from the first by copperplating one or more equipotential surfaces. Perhaps we should call the method "a solution in search of a problem." The situation was well described by Maxwell:

> "It appears, therefore, that what we should naturally call the inverse problem of determining the forms of the conductors when the expression for the potential is given is more manageable than the direct problem of determining the potential when the form of the conductors is given."[6]

If you worked Exercise 2.44, you already possess the raw material for an important example. You found that a uniform line charge of finite length has equipotential surfaces in the shape of prolate ellipsoids of revolution. This solves the problem of the potential and field of any isolated charged conductor of prolate spheroidal shape, reducing it to the relatively easy calculation of the potential due to a line charge. You can try it in Exercise 3.62.

3.5 Capacitance and capacitors

An isolated conductor carrying a charge Q has a certain potential ϕ_0, with zero potential at infinity; Q is proportional to ϕ_0. The constant of proportionality depends only on the size and shape of the conductor.

[6] See Maxwell (1891). Every student of physics ought sometime to look into Maxwell's book. Chapter VII is a good place to dip in while we are on the present subject. At the end of Volume I you will find some beautiful diagrams of electric fields, and shortly beyond the quotation we have just given, Maxwell's reason for presenting these figures. One may suspect that he also took delight in their construction and their elegance.

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We call this factor the *capacitance* of that conductor and denote it by C:

$$\boxed{Q = C\phi_0} \tag{3.7}$$

Obviously the units for C depend on the units in which Q and ϕ_0 are expressed. In our usual SI units, charge is measured in coulombs and potential in volts, so the capacitance C is measured in coulombs/volt. This combination of units is given its own name, the *farad:*

$$1 \text{ farad} = 1 \,\frac{\text{coulomb}}{\text{volt}}. \tag{3.8}$$

Since one volt equals one joule per coulomb, a farad can be expressed in terms of other units as[7]

$$1 \text{ farad} = 1 \,\frac{\text{C}^2\,\text{s}^2}{\text{kg m}^2}. \tag{3.9}$$

For an isolated spherical conductor of radius a we know that $\phi_0 = Q/4\pi\epsilon_0 a$. Hence the capacitance of the sphere, defined by Eq. (3.7), must be

$$C = \frac{Q}{\phi_0} = 4\pi\epsilon_0 a. \tag{3.10}$$

For an isolated conducting disk of radius a, according to Eq. (3.6), $Q = 8\epsilon_0 a\phi_0$, so the capacitance of such a conductor is $C = 8\epsilon_0 a$. It is somewhat smaller than the capacitance of a sphere of the same radius. In other words, the disk requires a smaller amount of charge to attain a given potential than does the sphere. This seems reasonable.

The farad happens to be a gigantic unit; the capacitance of an isolated sphere the size of the earth is only

$$\begin{aligned} C_\text{e} = 4\pi\epsilon_0 a &= 4\pi\left(8.85\cdot 10^{-12}\,\frac{\text{C}^2\,\text{s}^2}{\text{kg m}^3}\right)(6.4\cdot 10^6\text{ m}) \\ &\approx 7\cdot 10^{-4}\,\frac{\text{C}^2\,\text{s}^2}{\text{kg m}^2} = 7\cdot 10^{-4}\text{ farad}. \end{aligned} \tag{3.11}$$

But this causes no trouble. We deal on more familiar terms with the *microfarad* (μF), 10^{-6} farad, and the *picofarad* (pF), 10^{-12} farad. Note that the units of the constant ϵ_0 can be conveniently expressed as farads/meter. The capacitance will always involve one factor of ϵ_0 and one net power of length, so for conductors of a given shape, capacitance scales as a linear dimension of the object.

That applies to single, isolated conductors. The concept of capacitance is also useful whenever we are concerned with charges on and potentials of a general number of conductors. By far the most common

[7] In Gaussian units, Q is measured in esu and ϕ_0 in statvolts, so C is measured in esu/statvolt. Since in Gaussian units the esu can be written in terms of other fundamental units, you can show that the unit of capacitance is simply the centimeter, so it needs no other name.

case of interest is that of two conductors oppositely charged, with Q and $-Q$, respectively. Here the capacitance is defined as the ratio of the charge Q to the potential difference between the two conductors. The object itself, comprising the two conductors, insulating material to hold the conductors apart, and perhaps electrical terminals or leads, is called a *capacitor*. Most electronic circuits contain numerous capacitors. The parallel-plate capacitor is the simplest example.

Two similar flat conducting plates are arranged parallel to one another, separated by a distance s, as in Fig. 3.14(a). Let the area of each plate be A and suppose that there is a charge Q on one plate and $-Q$ on the other; ϕ_1 and ϕ_2 are the values of the potential at each of the plates. Figure 3.14(b) shows in cross section the field lines in this system. Away from the edge, the field is very nearly uniform in the region between the plates. When it is treated as uniform, its magnitude must be $(\phi_1 - \phi_2)/s$. The corresponding density of the surface charge on the inner surface of one of the plates is

$$\sigma = \epsilon_0 E = \frac{\epsilon_0(\phi_1 - \phi_2)}{s}. \tag{3.12}$$

If we may neglect the actual variation of E, and therefore of σ, which occurs principally near the edge of the plates, we can write a simple expression for the total charge, $Q = A\sigma$, on one plate:

$$Q = A\frac{\epsilon_0(\phi_1 - \phi_2)}{s} \qquad \text{(neglecting edge effects)}. \tag{3.13}$$

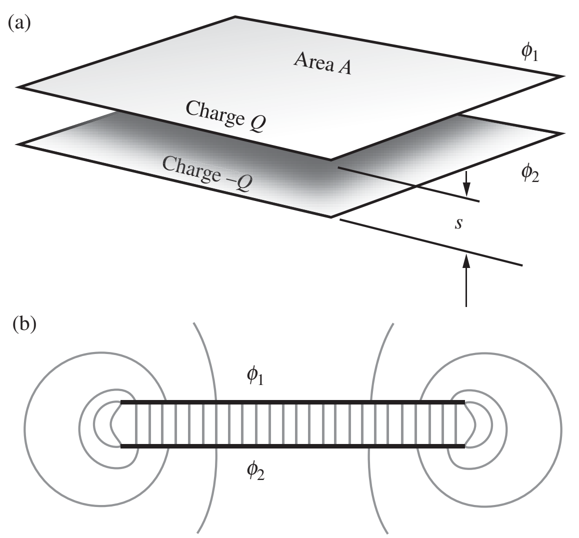

Figure 3.14.
(a) Parallel-plate capacitor. (b) Cross section of (a) showing field lines. The electric field is essentially uniform inside the capacitor.

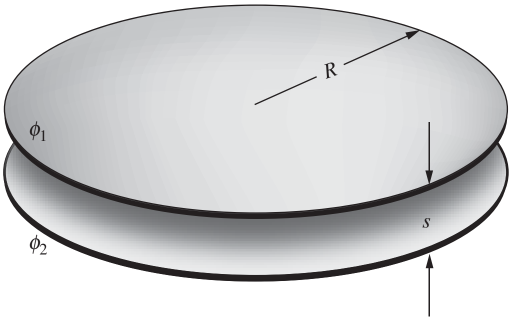

Figure 3.15.
The true capacitance of parallel circular plates, compared with the prediction of Eq. (3.13), for various ratios of separation to plate radius. The effect of the edge correction can be represented by writing the charge Q as

$$Q = \frac{\epsilon_0 A(\phi_1 - \phi_2)}{s} f.$$

For circular plates, the factor f depends on s/R as follows:

s/R	f
0.2	1.286
0.1	1.167
0.05	1.094
0.02	1.042
0.01	1.023

We should expect Eq. (3.13) to be more nearly accurate the smaller the ratio of the plate separation s to the lateral dimension of the plates. Of course, if we were to solve exactly the electrostatic problem, edge and all, for a particular shape of plate, we could replace Eq. (3.13) by an exact formula. To show how good an approximation Eq. (3.13) is, there are listed in Fig. 3.15 values of the correction factor f by which the charge Q given in Eq. (3.13) differs from the exact result, in the case of two conducting disks at various separations. The total charge is always a bit greater than Eq. (3.13) would predict. That seems reasonable as we look at Fig. 3.14(b), for there is evidently an extra concentration of charge at the edge, and even some charge on the outer surfaces near the edge.

We are not concerned now with the details of such corrections but with the general properties of a two-conductor system, the *capacitor*. We are interested in the relation between the charge Q on one of the plates and the potential difference between the two plates. For the particular system to which Eq. (3.13) applies, the quotient $Q/(\phi_1 - \phi_2)$ is $\epsilon_0 A/s$. Even if this is only approximate, it is clear that the exact formula will depend only on the size and geometrical arrangement of the plates. That is, for a fixed pair of conductors, the ratio of charge to potential difference will be a constant. We call this constant the *capacitance* of the capacitor and denote it usually by C.

$$Q = C(\phi_1 - \phi_2). \tag{3.14}$$

Thus the capacitance of the parallel-plate capacitor, with edge fields neglected, is given by

$$\boxed{C = \frac{\epsilon_0 A}{s}} \tag{3.15}$$

As with the above cases of the sphere and disk, this capacitance contains one factor of ϵ_0 and one net power of length. Figure 3.16 summarizes the formulas for capacitance in both SI and Gaussian units. Refer to it when in doubt. As usual, the differences stem from a factor $4\pi\epsilon_0$ in any expression involving charge. Appendix C gives the derivation that 1 cm (esu/statvolt) is equivalent to $1.11 \cdot 10^{-12}$ farad (coulomb/volt).

In defining the capacitance of a system of two conductors, we assume that their charges are equal and opposite (however, see the discussion below). This is a reasonable definition, because if we hook up a battery between two conductors that are initially neutral, then whatever charge leaves one of them ends up on the other. So when we talk about the "charge on the capacitor," we mean the charge on *either* of the conductors; the total charge of the system is zero, of course. Also, we define the capacitance to be a positive quantity. It will automatically come out to be positive if you remember that in Eq. (3.14) the charges Q and $-Q$ are associated with the potentials ϕ_1 and ϕ_2, respectively. But if you don't want to worry about the signs along the way, you can simply define the capacitance as $C = |Q|/|\phi_1 - \phi_2|$.

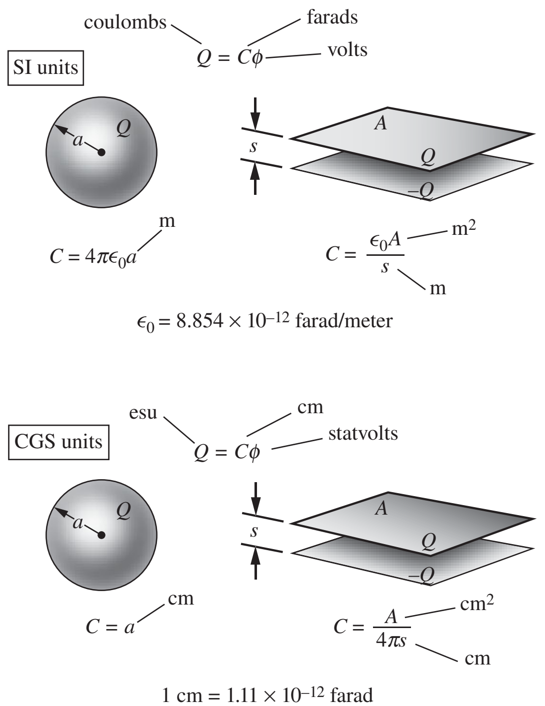

Figure 3.16.
Summary of units associated with capacitance.

Any pair of conductors, regardless of shape or arrangement, can be considered a capacitor. It just happens that the parallel-plate capacitor is a common arrangement and one for which an approximate calculation of the capacitance is very easy. Figure 3.17 shows two conductors, one inside the other. We can call this arrangement a capacitor too. As a practical matter, some mechanical support for the inner conductor would be needed, but that does not concern us. Also, to convey electric charge to or from the conductors we would need leads, which are themselves conducting bodies. Since a wire leading out from the inner body, numbered 1, necessarily crosses the space between the conductors, it is bound to cause some perturbation of the electric field in that space. To minimize this we may suppose the lead wires to be extremely thin, so that any charge residing on them is negligible. Or we might imagine the leads removed before the potentials are determined.

In this system we can distinguish three charges: Q_1, the total charge on the inner conductor; $Q_2^{(i)}$, the amount of charge on the inner surface of the outer conductor; $Q_2^{(o)}$, the charge on the outer surface of the outer conductor. Observe first that $Q_2^{(i)}$ must equal $-Q_1$. As we have seen in earlier examples, we know this because a surface such as S in Fig. 3.17 encloses both these charges and no others, and the flux through this surface is zero. The flux is zero because on the surface S, lying, as it does, in the interior of a conductor, the electric field is zero.

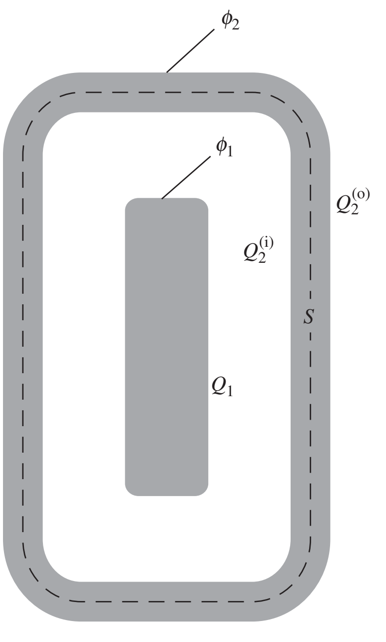

Figure 3.17.
A capacitor in which one conductor is enclosed by the other.

Evidently the value of Q_1 will uniquely determine the electric field within the region between the two conductors and thus will determine the difference between their potentials, $\phi_1 - \phi_2$. For that reason, if we are considering the two bodies as "plates" of a capacitor, it is only Q_1, or its counterpart $Q_2^{(i)}$, that is involved in determining the capacitance. The capacitance is given by

$$C = \frac{Q_1}{\phi_1 - \phi_2}. \tag{3.16}$$

$Q_2^{(o)}$ is here irrelevant, because piling more charge on the outer surface of the outer conductor increases both ϕ_1 and ϕ_2 by the *same* amount (because charge on a single conductor produces no electric field inside the conductor), thereby leaving the difference $\phi_1 - \phi_2$ unchanged. The complete enclosure of one conductor by the other makes the capacitance independent of everything outside. If you wish, you can consider this setup to be the superposition of the system consisting of the Q_1 and $Q_2^{(i)} = -Q_1$ conductors, plus the system consisting of the outer conductor containing an arbitrary charge $Q_2^{(o)}$ which doesn't affect the difference $\phi_1 - \phi_2$.

Example (Capacitance of two spherical shells) What is the capacitance of a capacitor that consists of two concentric spherical metal shells? The inner radius of the outer shell is a; the outer radius of the inner shell is b.

Solution Let there be charge Q on the inner shell and charge $-Q$ on the outer shell. As mentioned above, any additional charge on the outside surface of the outer shell doesn't affect the potential difference. The field between the shells is due only to the inner shell, so it equals $Q/4\pi\epsilon_0 r^2$. The magnitude of the potential difference is therefore

$$\Delta\phi = \int_b^a E\,dr = \int_b^a \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right). \tag{3.17}$$

The capacitance is then

$$C = \frac{Q}{\Delta\phi} = \frac{4\pi\epsilon_0}{\dfrac{1}{b} - \dfrac{1}{a}} = \frac{4\pi\epsilon_0 ab}{a - b}. \tag{3.18}$$

We can check this result by considering the limiting case where the gap between the conductors, $a - b$, is much smaller than b. In this limit the capacitor should be essentially the same as a flat-plate capacitor with separation $s = a - b$ and area $A = 4\pi r^2$, where $r \approx a \approx b$. And indeed, in this limit Eq. (3.18) gives $C \approx 4\pi\epsilon_0 r^2/s = \epsilon_0 A/s$, in agreement with Eq. (3.15). If we let r be the geometric mean of a and b, then the equivalence is exact, because the product ab in the numerator of C exactly equals r^2.

Also, in the $a \gg b$ limit, Eq. (3.18) gives $C = 4\pi\epsilon_0 b$, which is the correct result for the capacitance of an isolated sphere with radius b, with its counterpart at infinity; see Eq. (3.10).

3.6 Potentials and charges on several conductors

We have been skirting the edge of a more general problem, the relations among the charges and potentials of any number of conductors of some given configuration. The two-conductor capacitor is just a special case. It may surprise you that anything useful can be said about the general case. In tackling it, about all we can use is the uniqueness theorem and the superposition principle. To have something definite in mind, consider three separate conductors, all enclosed by a conducting shell, as in Fig. 3.18. The potential of this shell we may choose to be zero; with respect to this reference the potentials of the three conductors, for some particular state of the system, are ϕ_1, ϕ_2, and ϕ_3. The uniqueness theorem guarantees that, with ϕ_1, ϕ_2, and ϕ_3 given, the electric field is determined throughout the system. It follows that the charges Q_1, Q_2, and Q_3 on the individual conductors are likewise uniquely determined.

We need not keep account of the charge on the inner surface of the surrounding shell, since it will always be $-(Q_1 + Q_2 + Q_3)$. If you prefer, you can let "infinity" take over the role of this shell, imagining the shell to expand outward without limit. We have kept it in the picture because it makes the process of charge transfer easier to follow, for some people, if we have something to connect to.

Among the possible states of this system are ones with ϕ_2 and ϕ_3 both zero. We could enforce this condition by connecting conductors 2 and 3 to the zero-potential shell, as indicated in Fig. 3.18(a). As before, we may suppose the connecting wires are so thin that any charge residing on them is negligible. Of course, we really do not care how the specified condition is brought about. In such a state, which we shall call state I, the electric field in the whole system and the charge on every conductor is determined uniquely by the value of ϕ_1. Moreover, if ϕ_1 were doubled, that would imply a doubling of the field strength everywhere, and hence a doubling of each of the charges Q_1, Q_2, and Q_3. That is, with $\phi_2 = \phi_3 = 0$, each of the three charges must be proportional to ϕ_1. Stated mathematically:

- State I ($\phi_2 = \phi_3 = 0$):

$$Q_1 = C_{11}\phi_1; \quad Q_2 = C_{21}\phi_1; \quad Q_3 = C_{31}\phi_1. \tag{3.19}$$

The three constants, C_{11}, C_{21}, and C_{31}, can depend only on the shape and arrangement of the conducting bodies.

In just the same way we could analyze states in which ϕ_1 and ϕ_3 are zero, calling such a condition state II (Fig. 3.18(b)). Again, we find

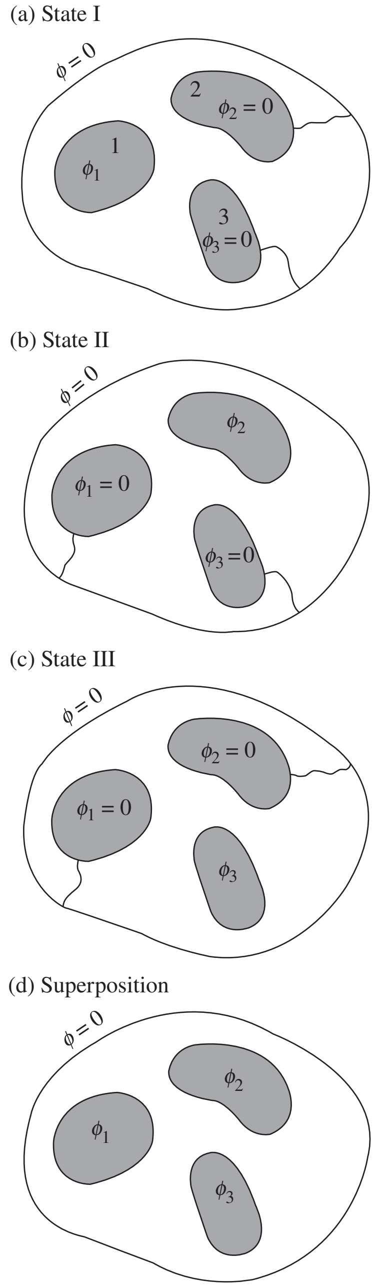

Figure 3.18.
A general state of this system can be analyzed as the superposition (d) of three states (a)–(c) in each of which all conductors but one are at zero potential.

a linear relation between the only nonzero potential, ϕ_2 in this case, and the various charges:

- State II ($\phi_1 = \phi_3 = 0$):

$$Q_1 = C_{12}\phi_2; \quad Q_2 = C_{22}\phi_2; \quad Q_3 = C_{32}\phi_2. \tag{3.20}$$

Finally, when ϕ_1 and ϕ_2 are held at zero, the field and the charges are proportional to ϕ_3:

- State III ($\phi_1 = \phi_2 = 0$):

$$Q_1 = C_{13}\phi_3; \quad Q_2 = C_{23}\phi_3; \quad Q_3 = C_{33}\phi_3. \tag{3.21}$$

Now the superposition of three states like I, II, and III is also a possible state. The electric field at any point is the vector sum of the electric fields at that point in the three cases, while the charge on a conductor is the sum of the charges it carried in the three cases. In this new state the potentials are ϕ_1, ϕ_2, and ϕ_3, none of them necessarily zero. In short, we have a completely general state. The relation connecting charges and potentials is obtained simply by adding Eqs. (3.19) through (3.21):

$$\begin{aligned} Q_1 &= C_{11}\phi_1 + C_{12}\phi_2 + C_{13}\phi_3, \\ Q_2 &= C_{21}\phi_1 + C_{22}\phi_2 + C_{23}\phi_3, \\ Q_3 &= C_{31}\phi_1 + C_{32}\phi_2 + C_{33}\phi_3. \end{aligned} \tag{3.22}$$

It appears that the electrical behavior of this system is characterized by the nine constants $C_{11}, C_{12}, \ldots, C_{33}$. In fact, only six constants are necessary, for it can be proved that in *any* system $C_{12} = C_{21}$, $C_{13} = C_{31}$, and $C_{23} = C_{32}$. Why this should be so is not obvious. Exercise 3.64 will suggest a proof based on conservation of energy, but for that purpose you will need an idea developed in Section 3.7. The C's in Eq. (3.22) are called the *coefficients of capacitance*. It is clear that our argument would extend to any number of conductors.

A set of equations like Eq. (3.22) can be solved for the ϕ's in terms of the Q's. That is, there is an equivalent set of linear relations of the form

$$\begin{aligned} \phi_1 &= P_{11}Q_1 + P_{12}Q_2 + P_{13}Q_3, \\ \phi_2 &= P_{21}Q_1 + P_{22}Q_2 + P_{23}Q_3, \\ \phi_3 &= P_{31}Q_1 + P_{32}Q_2 + P_{33}Q_3. \end{aligned} \tag{3.23}$$

The P's are called the *potential coefficients*; they could be computed from the C's, or vice versa.

We have here a simple example of the kind of relation we can expect to govern any *linear* physical system. Such relations turn up in the study of mechanical structures (connecting the strains with the loads), in the analysis of electrical circuits (connecting voltages and currents), and, generally speaking, wherever the superposition principle can be applied.

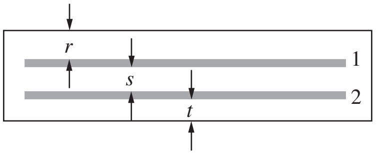

Figure 3.19.
Two capacitor plates inside a conducting box.

Example (Capacitance coefficients for two plates) Figure 3.19 shows in cross section a flat metal box in which there are two flat plates, 1 and 2, each of area A. The potential of the box is chosen to be zero. The various distances separating the plates from each other and from the top and bottom of the box, labeled r, s, and t in the figure, are to be assumed small compared with the width and length of the plates, so that it will be a good approximation to neglect the edge fields in estimating the charges on the plates. In this approximation, work out the capacitance coefficients, C_{11}, C_{22}, C_{12}, and C_{21}. Check that $C_{12} = C_{21}$.

Solution With the potential of the box chosen to be zero, we can write, in general,

$$Q_1 = C_{11}\phi_1 + C_{12}\phi_2,$$
$$Q_2 = C_{21}\phi_1 + C_{22}\phi_2. \tag{3.24}$$

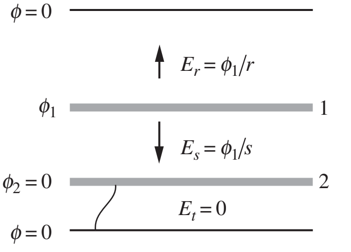

Figure 3.20.
The situation with the bottom plate grounded to the box.

Consider the case where ϕ_2 is made equal to zero by connecting plate 2 to the box. Then (see Fig. 3.20) the fields in the three regions are $E_r = \phi_1/r$, $E_s = \phi_1/s$, and $E_t = 0$. Gauss's law with a thin box completely surrounding plate 1 tells us that $Q_1 = \epsilon_0(AE_r + AE_s)$. Eliminating the E's in favor of the ϕ's gives

$$Q_1 = \epsilon_0 A\phi_1\left(\frac{1}{r} + \frac{1}{s}\right) \implies C_{11} = \epsilon_0 A\left(\frac{1}{r} + \frac{1}{s}\right). \tag{3.25}$$

Also, Gauss's law with a box around plate 2 tells us that $Q_2 = -\epsilon_0(AE_s + 0)$. Hence,

$$Q_2 = -\frac{\epsilon_0 A\phi_1}{s} \implies C_{21} = -\frac{\epsilon_0 A}{s}. \tag{3.26}$$

We can repeat the above arguments, but now with $\phi_1 = 0$ instead of $\phi_2 = 0$. This basically just involves switching the 1's and 2's, and letting $r \to t$ (but s remains s). We quickly find

$$C_{22} = \epsilon_0 A\left(\frac{1}{t} + \frac{1}{s}\right) \quad \text{and} \quad C_{12} = -\frac{\epsilon_0 A}{s}. \tag{3.27}$$

As expected, $C_{12} = C_{21}$. How do these four coefficients reduce to the $C = \epsilon_0 A/s$ capacitance we found for a parallel-plate capacitor in Eq. (3.15)? That is the subject of Problem 3.23.

3.7 Energy stored in a capacitor

Consider a capacitor of capacitance C, with a potential difference ϕ between the plates. The charge Q is equal to $C\phi$. There is a charge Q on one plate and $-Q$ on the other. Suppose we *increase* the charge from Q to $Q+dQ$ by transporting a positive charge dQ from the negative to the positive plate, working against the potential difference ϕ. The work that has to be done is $dW = \phi\, dQ = Q\, dQ/C$. Therefore to charge the capacitor starting from the uncharged state to some final charge Q_f requires the work

$$W = \frac{1}{C}\int_0^{Q_\text{f}} Q\, dQ = \frac{Q_\text{f}^2}{2C}. \tag{3.28}$$

This is the energy U that is "stored" in the capacitor. Since $Q_\text{f} = C\phi$, it can also be expressed by

$$\boxed{U = \frac{1}{2}C\phi^2} \tag{3.29}$$

where ϕ is the final potential difference between the plates. Using $Q = C\phi$ again, we can also write the energy as $U = Q\phi/2$. This result is consistent with the energy we would obtain from Eq. (2.32); see Exercise 3.65.

For the parallel-plate capacitor with plate area A and separation s, we found the capacitance $C = \epsilon_0 A/s$ and the electric field $E = \phi/s$. Hence Eq. (3.29) is also equivalent to

$$U = \frac{1}{2}\left(\frac{\epsilon_0 A}{s}\right)(Es)^2 = \frac{\epsilon_0 E^2}{2}\cdot As = \frac{\epsilon_0 E^2}{2}\cdot(\text{volume}). \tag{3.30}$$

This agrees with our general formula, Eq. (1.53), for the energy stored in an electric field.[8]

Equation (3.29) applies as well to the isolated charged conductor, which can be thought of as the inner plate of a capacitor, enclosed by an outer conductor of infinite size and potential zero. For the isolated sphere of radius a, we found $C = 4\pi\epsilon_0 a$, so that $U = (1/2)C\phi^2 = (1/2)(4\pi\epsilon_0 a)\phi^2$ or, equivalently, $U = (1/2)Q^2/C = (1/2)Q^2/4\pi\epsilon_0 a$, agreeing with the calculation in Problem 1.32 for the energy stored in the electric field of the charged sphere.

The oppositely charged plates of a capacitor will attract one another; some mechanical force will be required to hold them apart. This is obvious in the case of the parallel-plate capacitor, for which we could easily calculate the force on the surface charge. But we can make a more general statement based on Eq. (3.28), which relates stored energy to charge Q and capacitance C. Suppose that C depends in some manner on a linear coordinate x that measures the displacement of one "plate" of a capacitor, which might be a conductor of any shape, with respect to the other. Let F be the magnitude of the force that must be applied to each plate to overcome their attraction and keep x constant. Now imagine the distance x is increased by an increment Δx with Q remaining constant and one

[8] All this applies to the *vacuum capacitor* consisting of conductors with empty space in between. As you may know from the laboratory, most capacitors used in electric circuits are filled with an insulator or "dielectric." We are going to study the effect of that in Chapter 10.

plate fixed. The external force F on the other plate does work $F\,\Delta x$ and, if energy is to be conserved, this must appear as an increase in the stored energy $Q^2/2C$. That increase at constant Q is

$$\Delta U = \frac{dU}{dx}\,\Delta x = \frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right)\Delta x. \tag{3.31}$$

Equating this to the work $F\,\Delta x$ we find

$$F = \frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right). \tag{3.32}$$

Example (Parallel-plate capacitor) Let's verify that Eq. (3.32) yields the correct force on a plate in a parallel-plate capacitor. If the plate separation is x, Eq. (3.15) gives the capacitance as $C = \epsilon_0 A/x$. So Eq. (3.32) gives the (attractive) force as

$$F = \frac{Q^2}{2}\frac{d}{dx}\left(\frac{x}{\epsilon_0 A}\right) = \frac{Q^2}{2\epsilon_0 A}. \tag{3.33}$$

Is this correct? We know from Eq. (1.49) that the force (per unit area) on a sheet of charge equals the density σ times the average of the fields on either side. The total force on the entire plate of area A is then the total charge $Q = \sigma A$ times the average of the fields. The field is zero outside the capacitor, and it is σ/ϵ_0 inside. So the average of the two fields is $\sigma/2\epsilon_0$. (This is correctly the field due to the other plate, which is the field that the given plate feels.) The force on the plate is therefore

$$F = Q\frac{\sigma}{2\epsilon_0} = Q\frac{Q/A}{2\epsilon_0} = \frac{Q^2}{2\epsilon_0 A}, \tag{3.34}$$

as desired.

3.8 Other views of the boundary-value problem

It would be wrong to leave the impression that there are no general methods for dealing with the Laplacian boundary-value problem. Although we cannot pursue this question much further, we shall mention some useful and interesting approaches that you are likely to meet in future study of physics or applied mathematics.

First, an elegant method of analysis, called conformal mapping, is based on the theory of functions of a complex variable. Unfortunately it applies only to two-dimensional systems. These are systems in which ϕ depends only on x and y, for example, all conducting boundaries being cylinders (in the general sense) with elements running parallel to z. Laplace's equation then reduces to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{3.35}$$

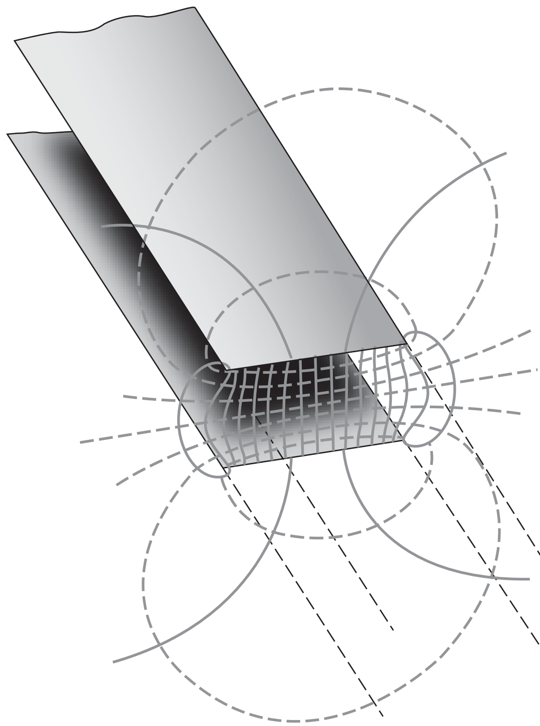

Figure 3.21.
Field lines and equipotentials for two infinitely long conducting strips.

with boundary values specified on some lines or curves in the *xy* plane. Many systems of practical interest are like this or sufficiently like this to make the method useful, quite apart from its intrinsic mathematical interest. For instance, the exact solution for the potential around two long parallel strips is easily obtained by the method of conformal mapping. The field lines and equipotentials are shown in a cross-sectional plane in Fig. 3.21. This provides us with the edge field for any parallel-plate capacitor in which the edge is long compared with the gap. The field shown in Fig. 3.14(b) was copied from such a solution. You will be able to apply this method after you have studied functions of a complex variable in more advanced mathematics courses.

Second, we mention a numerical method for finding approximate solutions of the electrostatic potential with given boundary values. Surprisingly simple and almost universally applicable, this method is based on that special property of harmonic functions with which we are already familiar: the value of the function at a point is equal to its average over the neighborhood of the point. In this method the potential function ϕ is represented by values at an array of discrete points only, including discrete points on the boundaries. The values at nonboundary points are then adjusted until each value is equal to the average of the neighboring values. In principle one could do this by solving a large number of simultaneous linear equations – as many as there are interior points.

But an approximate solution can be obtained by the following procedure, called a *relaxation method.* Start with the boundary points of the array, or grid, set at the values prescribed. Assign starting values arbitrarily to the interior points. Now visit, in some order, all the interior points. At each point reset its value to the average of the values at the four (for a square grid) adjacent grid points. Repeat again and again, until all the changes made in the course of one sweep over the network of interior points are acceptably small. If you want to see how this method works, Exercises 3.76 and 3.77 will provide an introduction. Whether convergence of the relaxation process can be ensured, or even hastened, and whether a relaxation method or direct solution of the simultaneous equations is the better strategy for a given problem, are questions in applied mathematics that we cannot go into here. It is the high-speed computer, of course, that makes both methods feasible.

3.9 Applications

The purpose of a *lightning rod* on a building is to provide an alternative path for the lightning's current on its way to ground, that is, a path that travels along a metal rod as opposed to through the building itself. Should the tip of the rod be pointed or rounded? The larger the field generated by the tip, the better the chance that a conductive path for the lightning is formed, meaning that the lightning is more likely to hit the rod than some other point on the building. On one hand, a pointed tip generates a large electric field very close to the tip, but on the other hand the field falls off more quickly than the field due to a more rounded tip (you can model the tip roughly as a small sphere). It isn't obvious which of these effects wins, but experiments suggest that a somewhat rounded tip has a better chance of being struck.

Capacitors have many uses; we will look at a few here. Capacitors can be used to store energy, for either slow discharge or fast discharge. In the slow case, the capacitor acts effectively like a battery. Examples include shake flashlights and power adapters. For the fast case, capacitors also have the ability to release their energy very quickly (unlike a normal battery). Examples include flashbulbs, stun guns, defibrillators, and the National Ignition Facility (NIF), whose goal is to create sustained fusion. The capacitor for a flashbulb might store 10 J of energy, while the huge capacitor bank at the NIF can store $4 \cdot 10^8$ J.

In many electronic devices, capacitors are used to smooth out fluctuations in the voltage in a DC circuit. If a capacitor is placed in parallel with the load, it acts like a reserve battery. If the voltage from the power supply dips, the capacitor will (temporarily) continue to push current through the load.

The *dynamic random access memory (DRAM)* in your computer works by storing charge on billions of tiny capacitors. Each capacitor represents a *bit* of information; uncharged is 0, charged is 1. However, the this surface. Where *is* the energy, then? Is it stored in the field, as Eq. 2.45 seems to suggest, or is it stored in the charge, as Eq. 2.43 implies? At the present stage this is simply an unanswerable question: I can tell you what the total energy is, and I can provide you with several different ways to compute it, but it is impertinent to worry about *where* the energy is located. In the context of radiation theory (Chapter 11) it is useful (and in general relativity it is *essential*) to regard the energy as stored in the field, with a density

$$\frac{\epsilon_0}{2}E^2 = \text{ energy per unit volume.} \tag{2.46}$$

But in electrostatics one could just as well say it is stored in the charge, with a density $\frac{1}{2}\rho V$. The difference is purely a matter of bookkeeping.

(iii) The superposition principle. Because electrostatic energy is *quadratic* in the fields, it does *not* obey a superposition principle. The energy of a compound system is *not* the sum of the energies of its parts considered separately—there are also "cross terms":

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2}\int E^2\,d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)^2\,d\tau \\ &= \frac{\epsilon_0}{2}\int \left(E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2\right)d\tau \\ &= W_1 + W_2 + \epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau. \end{aligned} \tag{2.47}$$

For example, if you double the charge everywhere, you *quadruple* the total energy.

Problem 2.36 Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.9.

Problem 2.37 Find the interaction energy ($\epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau$ in Eq. 2.47) for two point charges, q_1 and q_2, a distance a apart. [*Hint:* Put q_1 at the origin and q_2 on the z axis; use spherical coordinates, and do the r integral first.]

2.5 ■ CONDUCTORS

2.5.1 ■ Basic Properties

In an **insulator,** such as glass or rubber, each electron is on a short leash, attached to a particular atom. In a metallic **conductor,** by contrast, one or more electrons per atom are free to roam. (In liquid conductors such as salt water, it is ions that do the moving.) A *perfect* conductor would contain an *unlimited* supply of free charges. In real life there are no perfect conductors, but metals come pretty close, for most purposes.

From this definition, the basic electrostatic properties of ideal conductors immediately follow:

(i) $\mathbf{E} = 0$ inside a conductor. Why? Because if there *were* any field, those free charges would move, and it wouldn't be electro*statics* any more. Hmm ... that's hardly a satisfactory explanation; maybe all it proves is that you can't have electrostatics when conductors are present. We had better examine what happens when you put a conductor into an external electric field $\mathbf{E}_0$ (Fig. 2.42). Initially, the field will drive any free positive charges to the right, and negative ones to the left. (In practice, it's the negative charges—electrons—that do the moving, but when they depart, the right side is left with a net positive charge—the stationary nuclei—so it doesn't really matter which charges move; the effect is the same.) When they come to the edge of the material, the charges pile up: plus on the right side, minus on the left. Now, these **induced charges** produce a field of their own, $\mathbf{E}_1$, which, as you can see from the figure, is in the *opposite direction* to $\mathbf{E}_0$. That's the crucial point, for it means that the field of the induced charges *tends to cancel the original field.* Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero.[9] The whole process is practically instantaneous.

(ii) $\rho = 0$ inside a conductor. This follows from Gauss's law: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. If $\mathbf{E}$ is zero, so also is ρ. There is still charge around, but exactly as much plus as minus, so the *net* charge density in the interior is zero.

(iii) Any net charge resides on the surface. That's the only place left.

(iv) A conductor is an equipotential. For if $\mathbf{a}$ and $\mathbf{b}$ are any two points within (or at the surface of) a given conductor, $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$, and hence $V(\mathbf{a}) = V(\mathbf{b})$.

(v) $\mathbf{E}$ is perpendicular to the surface, just outside a conductor. Otherwise, as in **(i),** charge will immediately flow around the surface until it kills off the tangential component (Fig. 2.43). (*Perpendicular* to the surface, charge cannot flow, of course, since it is confined to the conducting object.)

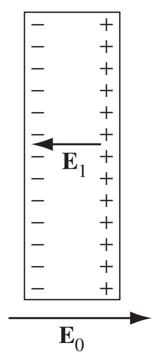

FIGURE 2.42

[9] *Outside* the conductor the field is *not* zero, for here $\mathbf{E}_0$ and $\mathbf{E}_1$ do *not* tend to cancel.

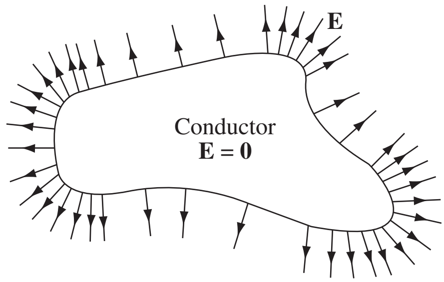

FIGURE 2.43

I think it is astonishing that the charge on a conductor flows to the surface. Because of their mutual repulsion, the charges naturally spread out as much as possible, but for *all* of them to go to the surface seems like a waste of the interior space. Surely we could do better, from the point of view of making each charge as far as possible from its neighbors, to sprinkle *some* of them throughout the volume . . . Well, it simply is not so. You do best to put *all* the charge on the surface, and this is true regardless of the size or shape of the conductor.[10]

The problem can also be phrased in terms of energy. Like any other free dynamical system, the charge on a conductor will seek the configuration that minimizes its potential energy. What property **(iii)** asserts is that the electrostatic energy of a solid object (with specified shape and total charge) is a minimum when that charge is spread over the surface. For instance, the energy of a sphere is $(1/8\pi\epsilon_0)(q^2/R)$ if the charge is uniformly distributed over the surface, as we found in Ex. 2.9, but it is greater, $(3/20\pi\epsilon_0)(q^2/R)$, if the charge is uniformly distributed throughout the volume (Prob. 2.34).

2.5.2 ■ Induced Charges

If you hold a charge $+q$ near an uncharged conductor (Fig. 2.44), the two will attract one another. The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side. (Another way to think of it is that the charge moves around in such a way as to kill off the field of q for points inside the conductor, where the total field must be zero.) Since the negative induced charge is closer to q, there is a net force of attraction. (In Chapter 3 we shall calculate this force explicitly, for the case of a spherical conductor.)

When I speak of the field, charge, or potential "inside" a conductor, I mean in the "meat" of the conductor; if there is some hollow *cavity* in the conductor, and

[10]By the way, the one- and two-dimensional analogs are quite different: The charge on a conducting *disk* does *not* all go to the perimeter (R. Friedberg, *Am. J. Phys.* **61**, 1084 (1993)), nor does the charge on a conducting needle go to the ends (D. J. Griffiths and Y. Li, *Am. J. Phys.* **64**, 706 (1996))—see Prob. 2.57. Moreover, if the exponent of r in Coulomb's law were not precisely 2, the charge on a solid conductor would not all go to the surface—see D. J. Griffiths and D. Z. Uvanovic, *Am. J. Phys.* **69**, 435 (2001), and Prob. 2.54g.

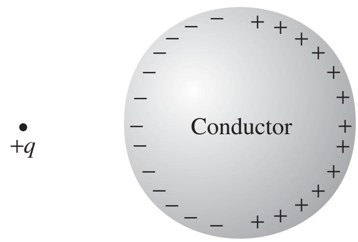

FIGURE 2.44

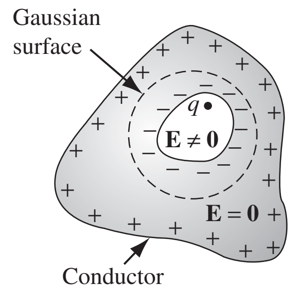

FIGURE 2.45

within that cavity you put some charge, then the field *in the cavity* will *not* be zero. But in a remarkable way the cavity and its contents are electrically isolated from the outside world by the surrounding conductor (Fig. 2.45). No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge there. Similarly, the field due to charges within the cavity is canceled, for all exterior points, by the induced charge on the inner surface. However, the compensating charge left over on the *outer* surface of the conductor effectively "communicates" the presence of q to the outside world. The total charge induced on the cavity wall is equal and opposite to the charge inside, for if we surround the cavity with a Gaussian surface, all points of which are in the conductor (Fig. 2.45), $\oint \mathbf{E} \cdot d\mathbf{a} = 0$, and hence (by Gauss's law) the net enclosed charge must be zero. But $Q_{\text{enc}} = q + q_{\text{induced}}$, so $q_{\text{induced}} = -q$. Then if the conductor as a whole is electrically neutral, there must be a charge $+q$ on its outer surface.

Example 2.10. An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge q. *Question:* What is the field outside the sphere?

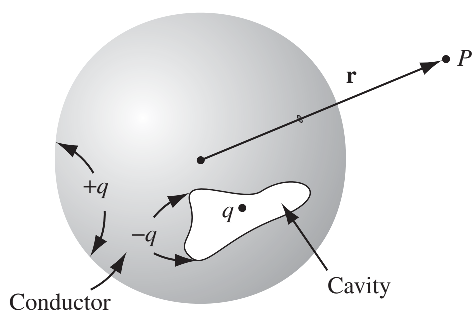

FIGURE 2.46

Solution

At first glance, it would appear that the answer depends on the shape of the cavity and the location of the charge. But that's wrong: the answer is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

regardless. The conductor conceals from us all information concerning the nature of the cavity, revealing only the total charge it contains. How can this be? Well, the charge $+q$ induces an opposite charge $-q$ on the wall of the cavity, which distributes itself in such a way that its field cancels that of q, for all points exterior to the cavity. Since the conductor carries no net charge, this leaves $+q$ to distribute itself uniformly over the surface of the sphere. (It's *uniform* because the asymmetrical influence of the point charge $+q$ is negated by that of the induced charge $-q$ on the inner surface.) For points outside the sphere, then, the only thing that survives is the field of the leftover $+q$, uniformly distributed over the outer surface.

It may occur to you that in one respect this argument is open to challenge: There are actually *three* fields at work here: $\mathbf{E}_q$, $\mathbf{E}_{\text{induced}}$, and $\mathbf{E}_{\text{leftover}}$. All we know for certain is that the sum of the three is zero inside the conductor, yet I claimed that the first two *alone* cancel, while the third is separately zero there. Moreover, even if the first two cancel within the conductor, who is to say they still cancel for points outside? They do not, after all, cancel for points *inside* the cavity. I cannot give you a completely satisfactory answer at the moment, but this much at least is true: There *exists* a way of distributing $-q$ over the inner surface so as to cancel the field of q at all exterior points. For that same cavity could have been carved out of a *huge* spherical conductor with a radius of 27 miles or light years or whatever. In that case, the leftover $+q$ on the outer surface is simply too far away to produce a significant field, and the other two fields would *have* to accomplish the cancellation by themselves. So we know they *can* do it ... but are we sure they *choose* to? Perhaps for small spheres nature prefers some complicated three-way cancellation. Nope: As we'll see in the uniqueness theorems of Chapter 3, electrostatics is very stingy with its options; there is always precisely one way—no more—of distributing the charge on a conductor so as to make the field inside zero. Having found a *possible* way, we are guaranteed that no alternative exists, even in principle.

If a cavity surrounded by conducting material is itself empty of charge, then the field within the cavity is zero. For any field line would have to begin and end on the cavity wall, going from a plus charge to a minus charge (Fig. 2.47). Letting that field line be part of a closed loop, the rest of which is entirely inside the conductor (where $\mathbf{E} = \mathbf{0}$), the integral $\oint \mathbf{E} \cdot d\mathbf{l}$ is distinctly *positive,* in violation of Eq. 2.19. It follows that $\mathbf{E} = \mathbf{0}$ within an *empty* cavity, and there is in fact *no* charge on the surface of the cavity. (This is why you are relatively safe inside a metal car during a thunderstorm—you may get *cooked,* if lightning strikes, but you will not be *electrocuted.* The same principle applies to the placement of sensitive apparatus

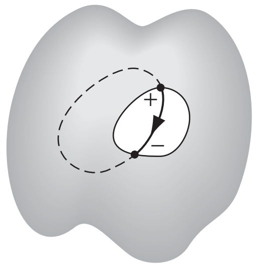

FIGURE 2.47

inside a grounded **Faraday cage,** to shield out stray electric fields. In practice, the enclosure doesn't even have to be solid conductor—chicken wire will often suffice.)

Problem 2.38 A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in Fig. 2.48). The shell carries no net charge.

(a) Find the surface charge density σ at R, at a, and at b.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

Problem 2.39 Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed—call these charges q_a and q_b.

(a) Find the surface charge densities σ_a, σ_b, and σ_R.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

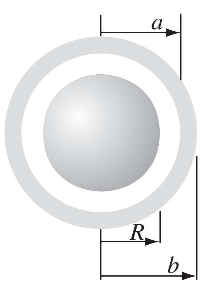

FIGURE 2.48

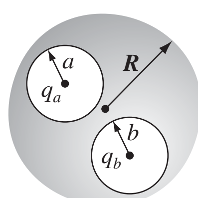

FIGURE 2.49

(e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

Problem 2.40

(a) A point charge q is inside a cavity in an uncharged conductor (Fig. 2.45). Is the force on q necessarily zero?[11]

(b) Is the force between a point charge and a nearby uncharged conductor always attractive?[12]

2.5.3 ■ Surface Charge and the Force on a Conductor

Because the field inside a conductor is zero, boundary condition 2.33 requires that the field immediately *outside* is

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}, \tag{2.48}$$

consistent with our earlier conclusion that the field is normal to the surface. In terms of potential, Eq. 2.36 yields

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}. \tag{2.49}$$

These equations enable you to calculate the surface charge on a conductor, if you can determine $\mathbf{E}$ or V; we shall use them frequently in the next chapter.

In the presence of an electric field, a surface charge will experience a force; the force per unit area, $\mathbf{f}$, is $\sigma\mathbf{E}$. But there's a problem here, for the electric field is *discontinuous* at a surface charge, so what are we supposed to use: $\mathbf{E}_{\text{above}}$, $\mathbf{E}_{\text{below}}$, or something in between? The answer is that we should use the *average* of the two:

$$\mathbf{f} = \sigma\mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}). \tag{2.50}$$

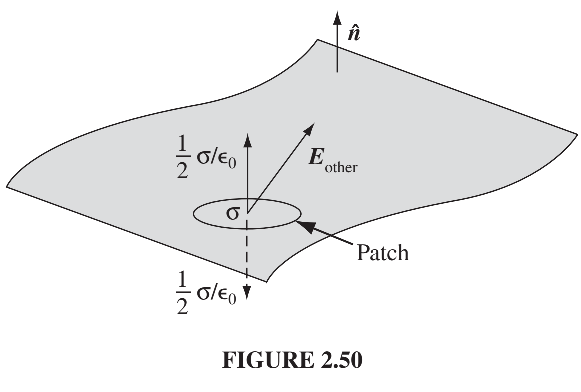

FIGURE 2.50

[11] This problem was suggested by Nelson Christensen.
[12] See M. Levin and S. G. Johnson, *Am. J. Phys.* **79**, 843 (2011).

Why the average? The reason is very simple, though the telling makes it sound complicated: Let's focus our attention on a tiny patch of surface surrounding the point in question (Fig. 2.50). (Make it small enough so it is essentially flat and the surface charge on it is essentially constant.) The *total* field consists of two parts—that attributable to the patch itself, and that due to everything else (other regions of the surface, as well as any external sources that may be present):

$$\mathbf{E} = \mathbf{E}_{\text{patch}} + \mathbf{E}_{\text{other}}.$$

Now, the patch cannot exert a force on itself, any more than you can lift yourself by standing in a basket and pulling up on the handles. The force on the patch, then, is due exclusively to $\mathbf{E}_{\text{other}}$, and *this* suffers *no* discontinuity (if we removed the patch, the field in the "hole" would be perfectly smooth). The discontinuity is due entirely to the charge on the patch, which puts out a field $(\sigma/2\epsilon_0)$ on either side, pointing away from the surface. Thus,

$$\mathbf{E}_{\text{above}} = \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}},$$

$$\mathbf{E}_{\text{below}} = \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}},$$

and hence

$$\mathbf{E}_{\text{other}} = \frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}) = \mathbf{E}_{\text{average}}.$$

Averaging is really just a device for removing the contribution of the patch itself.

That argument applies to *any* surface charge; in the particular case of a conductor, the field is zero inside and $(\sigma/\epsilon_0)\hat{\mathbf{n}}$ outside (Eq. 2.48), so the average is $(\sigma/2\epsilon_0)\hat{\mathbf{n}}$, and the force per unit area is

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}. \tag{2.51}$$

This amounts to an outward **electrostatic pressure** on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field just outside the surface,

$$P = \frac{\epsilon_0}{2}E^2. \tag{2.52}$$

Problem 2.41 Two large metal plates (each of area A) are held a small distance d apart. Suppose we put a charge Q on each plate; what is the electrostatic pressure on the plates?

Problem 2.42 A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

FIGURE 2.51

2.5.4 ■ Capacitors

Suppose we have *two* conductors, and we put charge $+Q$ on one and $-Q$ on the other (Fig. 2.51). Since V is constant over a conductor, we can speak unambiguously of the potential difference between them:

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}.$$

We don't know how the charge distributes itself over the two conductors, and calculating the field would be a nightmare, if their shapes are complicated, but this much we *do* know: **E** is *proportional* to Q. For **E** is given by Coulomb's law:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau,$$

so if you double ρ, you double **E**. [Wait a minute! How do we know that doubling Q (and also $-Q$) simply doubles ρ? Maybe the charge *moves around* into a completely different configuration, quadrupling ρ in some places and halving it in others, just so the *total* charge on each conductor is doubled. The *fact* is that this concern is unwarranted—doubling Q *does* double ρ everywhere; it *doesn't* shift the charge around. The proof of this will come in Chapter 3; for now you'll just have to trust me.]

Since **E** is proportional to Q, so also is V. The constant of proportionality is called the **capacitance** of the arrangement:

$$C \equiv \frac{Q}{V}. \tag{2.53}$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, C is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad (10^{-6} F) and the picofarad (10^{-12} F).

Notice that V is, by definition, the potential of the *positive* conductor less that of the negative one; likewise, Q is the charge of the *positive* conductor. Accordingly, capacitance is an intrinsically positive quantity. (By the way, you will occasionally hear someone speak of the capacitance of a *single* conductor. In this case the "second conductor," with the negative charge, is an imaginary spherical shell of infinite radius surrounding the one conductor. It contributes nothing to the field, so the capacitance is given by Eq. 2.53, where V is the potential with infinity as the reference point.)

the correct value on the boundaries, then it's *right*. You'll see the power of this argument when we come to the method of images.

Incidentally, it is easy to improve on the first uniqueness theorem: I assumed there was no charge inside the region in question, so the potential obeyed Laplace's equation, but we may as well throw in some charge (in which case V obeys Poisson's equation). The argument is the same, only this time

$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho, \qquad \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho,$$

so

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0.$$

Once again the *difference* ($V_3 \equiv V_1 - V_2$) satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence $V_1 = V_2$.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

3.1.6 ■ Conductors and the Second Uniqueness Theorem

The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of V on all surfaces surrounding the region of interest. And this situation often occurs in practice: In the laboratory, we have conductors connected to batteries, which maintain a given potential, or to **ground**, which is the experimentalist's word for $V = 0$. However, there are other circumstances in which we do not know the *potential* at the boundary, but rather the *charges* on various conducting surfaces. Suppose I put charge Q_a on the first conductor, Q_b on the second, and so on—I'm not telling you how the charge distributes itself over each conducting surface, because as soon as I put it on, it moves around in a way I do not control. And for good measure, let's say there is some specified charge density ρ in the region between the conductors. Is the electric field now uniquely determined? Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given (Fig. 3.6). (The region as a whole can be bounded by another conductor, or else unbounded.)

Proof. Suppose there are *two* fields satisfying the conditions of the problem. Both obey Gauss's law in differential form in the space between the conductors:

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

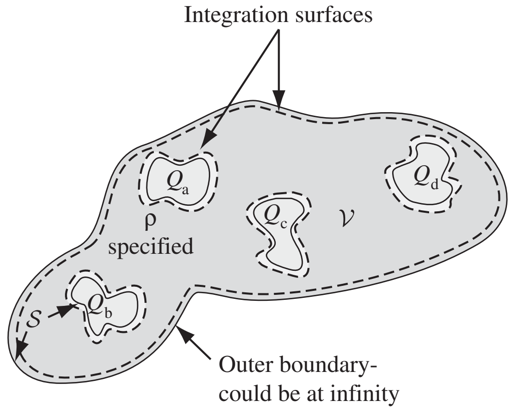

FIGURE 3.6

And both obey Gauss's law in integral form for a Gaussian surface enclosing each conductor:

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i, \qquad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i.$$

Likewise, for the outer boundary (whether this is just inside an enclosing conductor or at infinity),

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}.$$

As before, we examine the difference

$$\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2,$$

which obeys

$$\nabla \cdot \mathbf{E}_3 = 0 \tag{3.7}$$

in the region between the conductors, and

$$\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \tag{3.8}$$

over each boundary surface.

Now there is one final piece of information we must exploit: Although we do not know how the charge Q_i distributes itself over the ith conductor, we *do* know that each conductor is an equipotential, and hence V_3 is a *constant* (not necessarily the *same* constant) over each conducting surface. (It need not be *zero,* for the potentials V_1 and V_2 may not be equal—all we know for sure is that *both* are *constant* over any given conductor.) Next comes a trick. Invoking product rule number 5 (inside front cover), we find that

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2.$$

Here I have used Eq. 3.7, and $\mathbf{E}_3 = -\nabla V_3$. Integrating this over $\mathcal{V}$, and applying the divergence theorem to the left side:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau.$$

The surface integral covers all boundaries of the region in question—the conductors and outer boundary. Now V_3 is a constant over each surface (if the outer boundary is infinity, $V_3 = 0$ there), so it comes outside each integral, and what remains is zero, according to Eq. 3.8. Therefore,

$$\int_{\mathcal{V}} (E_3)^2\, d\tau = 0.$$

But this integrand is never negative; the only way the integral can vanish is if $E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$, and the theorem is proved. □

This proof was not easy, and there is a real danger that the theorem itself will seem more plausible to you than the proof. In case you think the second uniqueness theorem is "obvious," consider this example of Purcell's: Figure 3.7 shows a simple electrostatic configuration, consisting of four conductors with charges $\pm Q$, situated so that the plusses are near the minuses. It all looks very comfortable. Now, what happens if we join them in pairs, by tiny wires, as indicated in Fig. 3.8? Since the positive charges are very near negative charges (which is where they *like* to be) you might well guess that *nothing* will happen—the configuration looks stable.

Well, that sounds reasonable, but it's wrong. The configuration in Fig. 3.8 is *impossible.* For there are now effectively *two* conductors, and the total charge on each is *zero*. *One* possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere, and hence zero field

FIGURE 3.7

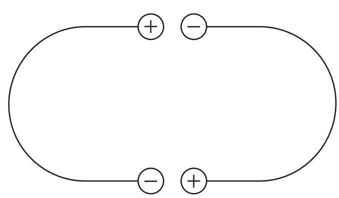

FIGURE 3.8

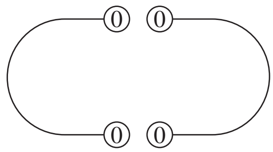

FIGURE 3.9

everywhere (Fig. 3.9). By the second uniqueness theorem, this must be *the* solution: The charge will flow down the tiny wires, canceling itself off.

Problem 3.5 Prove that the field is uniquely determined when the charge density ρ is given and *either* V *or* the normal derivative $\partial V/\partial n$ is specified on each boundary surface. Do not assume the boundaries are conductors, or that V is constant over any given surface.

Problem 3.6 A more elegant proof of the second uniqueness theorem uses Green's identity (Prob. 1.61c), with $T = U = V_3$. Supply the details.

3.2 ■ THE METHOD OF IMAGES

3.2.1 ■ The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane (Fig. 3.10). *Question:* What is the potential in the region above the plane? It's not just $(1/4\pi\epsilon_0)q/\mathscr{r}$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge. But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

From a mathematical point of view, our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge $\big($that is, for $x^2 + y^2 + z^2 \gg d^2\big)$.

The first uniqueness theorem (actually, its corollary) guarantees that there is only one function that meets these requirements. If by trick or clever guess we can discover such a function, it's got to be the answer.

Trick: Forget about the actual problem; we're going to study a *completely different* situation. This new configuration consists of *two* point charges, $+q$ at

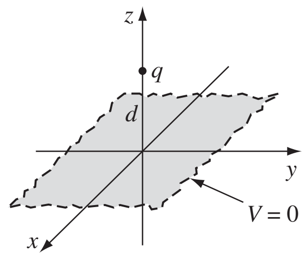

FIGURE 3.10

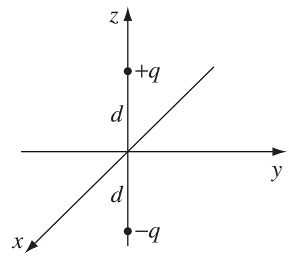

FIGURE 3.11

(0, 0, d) and $-q$ at (0, 0, $-d$), and *no* conducting plane (Fig. 3.11). For this configuration, I can easily write down the potential:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]. \quad (3.9)$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$. But these are precisely the conditions of the original problem! Evidently the second configuration happens to produce exactly the same potential as the first configuration, in the "upper" region $z \geq 0$. (The "lower" region, $z < 0$, is completely different, but who cares? The upper part is all we need.) *Conclusion:* The potential of a point charge above an infinite grounded conductor is given by Eq. 3.9, for $z \geq 0$.

Notice the crucial role played by the uniqueness theorem in this argument: without it, no one would believe this solution, since it was obtained for a completely different charge distribution. But the uniqueness theorem certifies it: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

3.2.2 ■ Induced Surface Charge

Now that we know the potential, it is a straightforward matter to compute the surface charge σ induced on the conductor. According to Eq. 2.49,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n},$$

where $\partial V/\partial n$ is the normal derivative of V at the surface. In this case the normal direction is the z direction, so

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}.$$

From Eq. 3.9,

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\},$$

so[5]

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \tag{3.10}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

While we're at it, let's compute the *total* induced charge

$$Q = \int \sigma \, da.$$

This integral, over the xy plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but it's a little easier to use polar coordinates (r, ϕ), with $r^2 = x^2+y^2$ and $da = r\,dr\,d\phi$. Then

$$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}},$$

and

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\,dr\,d\phi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q. \tag{3.11}$$

The total charge induced on the plane is $-q$, as (with benefit of hindsight) you can perhaps convince yourself it *had* to be.

3.2.3 ■ Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge. Let's calculate the force of attraction. Since the potential in the vicinity of q is the same as in the analog problem (the one with $+q$ and $-q$ but no conductor), so also is the field and, therefore, the force:

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}. \tag{3.12}$$

[5]For an entirely different derivation of this result, see Prob. 3.38.

Beware: It is easy to get carried away, and assume that *everything* is the same in the two problems. Energy, however, is *not* the same. With the two point charges and no conductor, Eq. 2.42 gives

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}. \tag{3.13}$$

But for a single charge and conducting plane, the energy is *half* of this:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \tag{3.14}$$

Why half? Think of the energy stored in the fields (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau.$$

In the first case, both the upper region ($z > 0$) and the lower region ($z < 0$) contribute—and by symmetry they contribute equally. But in the second case, only the upper region contains a nonzero field, and hence the energy is half as great.[6]

Of course, one could also determine the energy by calculating the work required to bring q in from infinity. The force required (to oppose the electrical force in Eq. 3.12) is $(1/4\pi\epsilon_0)(q^2/4z^2)\hat{\mathbf{z}}$, so

$$\begin{aligned} W &= \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,dz \\ &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \end{aligned}$$

As I move q toward the conductor, I do work *only on q*. It is true that induced charge is moving in over the conductor, but this costs me nothing, since the whole conductor is at potential zero. By contrast, if I simultaneously bring in *two* point charges (with no conductor), I do work on *both* of them, and the total is (again) twice as great.

3.2.4 ■ Other Image Problems

The method just described is not limited to a single point charge; *any* stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image—hence the name **method of images.** (Remember that the image charges have the *opposite sign;* this is what guarantees that the xy plane will be at potential zero.) There are also some exotic problems that can be handled in similar fashion; the nicest of these is the following.

[6]For a generalization of this result, see M. M. Taddei, T. N. C. Mendes, and C. Farina, *Eur. J. Phys.* **30**, 965 (2009), and Prob. 3.41b.

Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.

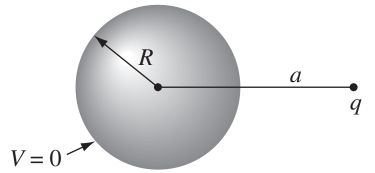

FIGURE 3.12

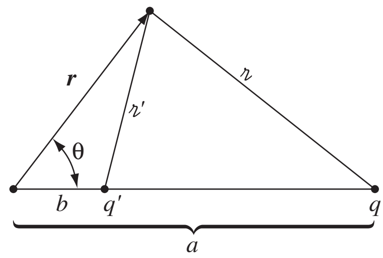

FIGURE 3.13

Solution

Examine the *completely different* configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q, \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

to the right of the center of the sphere (Fig. 3.13). No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right), \tag{3.17}$$

where $\mathscr{r}$ and $\mathscr{r}'$ are the distances from q and q', respectively. Now, it happens (see Prob. 3.8) that this potential vanishes at all points on the sphere, and therefore fits the boundary conditions for our original problem, in the exterior region.[7]

Conclusion: Eq. 3.17 is the potential of a point charge near a grounded conducting sphere. (Notice that b is less than R, so the "image" charge q' is safely inside the sphere—*you cannot put image charges in the region where you are calculating V*; that would change ρ, and you'd be solving Poisson's equation with

[7]This solution is due to William Thomson (later Lord Kelvin), who published it in 1848, when he was just 24. It was apparently inspired by a theorem of Apollonius (200 BC) that says the locus of points with a fixed ratio of distances from two given points is a sphere. See J. C. Maxwell, "Treatise on Electricity and Magnetism, Vol. I," Dover, New York, p. 245. I thank Gabriel Karl for this interesting history.

the wrong source.) In particular, the force of attraction between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}. \tag{3.18}$$

The method of images is delightfully simple ... when it works. But it is as much an art as a science, for you must somehow think up just the right "auxiliary" configuration, and for most shapes this is forbiddingly complicated, if not impossible.

Problem 3.7 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)

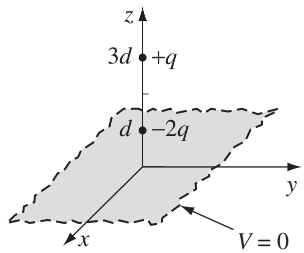

FIGURE 3.14

Problem 3.8

(a) Using the law of cosines, show that Eq. 3.17 can be written as follows:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right], \tag{3.19}$$

where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form, it is obvious that $V = 0$ on the sphere, $r = R$.

(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

(c) Calculate the energy of this configuration.

Problem 3.9 In Ex. 3.2 we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential V_0 (relative, of course, to infinity). What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a *neutral* conducting sphere.

! **Problem 3.10** A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

(a) Find the potential in the region above the plane. [*Hint:* Refer to Prob. 2.52.]

(b) Find the charge density σ induced on the conducting plane.

Problem 3.11 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

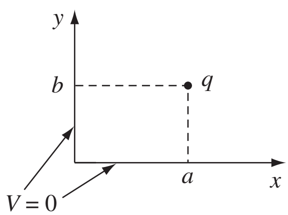

FIGURE 3.15

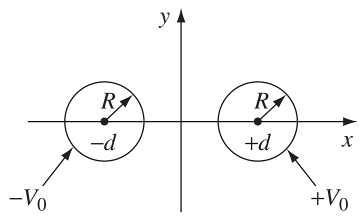

FIGURE 3.16

! **Problem 3.12** Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart. One is at potential V_0, the other at $-V_0$ (Fig. 3.16). Find the potential everywhere. [*Hint:* Exploit the result of Prob. 2.52.]

3.3 ■ SEPARATION OF VARIABLES

In this section we shall attack Laplace's equation directly, using the method of **separation of variables**, which is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: *We look for solutions that are* products *of functions, each of which depends on only* one *of the coordinates*. The algebraic details, however, can be formidable, so I'm going to develop the method through a sequence of examples. We'll start with Cartesian coordinates and then do spherical coordinates (I'll leave the cylindrical case for you to tackle on your own, in Prob. 3.24).

FIGURE 2.51

2.5.4 ■ Capacitors

Suppose we have *two* conductors, and we put charge $+Q$ on one and $-Q$ on the other (Fig. 2.51). Since V is constant over a conductor, we can speak unambiguously of the potential difference between them:

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}.$$

We don't know how the charge distributes itself over the two conductors, and calculating the field would be a nightmare, if their shapes are complicated, but this much we *do* know: **E** is *proportional* to Q. For **E** is given by Coulomb's law:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau,$$

so if you double ρ, you double **E**. [Wait a minute! How do we know that doubling Q (and also $-Q$) simply doubles ρ? Maybe the charge *moves around* into a completely different configuration, quadrupling ρ in some places and halving it in others, just so the *total* charge on each conductor is doubled. The *fact* is that this concern is unwarranted—doubling Q *does* double ρ everywhere; it *doesn't* shift the charge around. The proof of this will come in Chapter 3; for now you'll just have to trust me.]

Since **E** is proportional to Q, so also is V. The constant of proportionality is called the **capacitance** of the arrangement:

$$C \equiv \frac{Q}{V}. \tag{2.53}$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, C is measured in **farads** (F); a farad is a coulomb-per-volt. Actually, this turns out to be inconveniently large; more practical units are the microfarad (10^{-6} F) and the picofarad (10^{-12} F).

Notice that V is, by definition, the potential of the *positive* conductor less that of the negative one; likewise, Q is the charge of the *positive* conductor. Accordingly, capacitance is an intrinsically positive quantity. (By the way, you will occasionally hear someone speak of the capacitance of a *single* conductor. In this case the "second conductor," with the negative charge, is an imaginary spherical shell of infinite radius surrounding the one conductor. It contributes nothing to the field, so the capacitance is given by Eq. 2.53, where V is the potential with infinity as the reference point.)

Example 2.11. Find the capacitance of a **parallel-plate capacitor** consisting of two metal surfaces of area A held a distance d apart (Fig. 2.52).

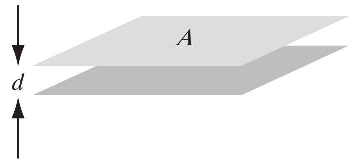

FIGURE 2.52

Solution

If we put $+Q$ on the top and $-Q$ on the bottom, they will spread out uniformly over the two surfaces, provided the area is reasonably large and the separation small.[13] The surface charge density, then, is $\sigma = Q/A$ on the top plate, and so the field, according to Ex. 2.6, is $(1/\epsilon_0)Q/A$. The potential difference between the plates is therefore

$$V = \frac{Q}{A\epsilon_0}d,$$

and hence

$$C = \frac{A\epsilon_0}{d}. \tag{2.54}$$

If, for instance, the plates are square with sides 1 cm long, and they are held 1 mm apart, then the capacitance is 9×10^{-13} F.

Example 2.12. Find the capacitance of two concentric spherical metal shells, with radii a and b.

Solution

Place charge $+Q$ on the inner sphere, and $-Q$ on the outer one. The field between the spheres is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}},$$

so the potential difference between them is

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{4\pi\epsilon_0}\int_b^a \frac{1}{r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

[13]The *exact* solution is not easy—even for the simpler case of circular plates. See G. T. Carlson and B. L. Illman, *Am. J. Phys.* **62**, 1099 (1994).

As promised, V is proportional to Q; the capacitance is

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{(b-a)}.$$

To "charge up" a capacitor, you have to remove electrons from the positive plate and carry them to the negative plate. In doing so, you fight against the electric field, which is pulling them back toward the positive conductor and pushing them away from the negative one. How much work does it take, then, to charge the capacitor up to a final amount Q? Suppose that at some intermediate stage in the process the charge on the positive plate is q, so that the potential difference is q/C. According to Eq. 2.38, the work you must do to transport the next piece of charge, dq, is

$$dW = \left(\frac{q}{C}\right) dq.$$

The total work necessary, then, to go from $q = 0$ to $q = Q$, is

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C},$$

or, since $Q = CV$,

$$W = \frac{1}{2}CV^2, \qquad (2.55)$$

where V is the final potential of the capacitor.

Problem 2.43 Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig. 2.53).

FIGURE 2.53

Problem 2.44 Suppose the plates of a parallel-plate capacitor move closer together by an infinitesimal distance ϵ, as a result of their mutual attraction.

(a) Use Eq. 2.52 to express the work done by electrostatic forces, in terms of the field E, and the area of the plates, A.

(b) Use Eq. 2.46 to express the energy lost by the field in this process.

(This problem is supposed to be easy, but it contains the embryo of an alternative derivation of Eq. 2.52, using conservation of energy.)

More Problems on Chapter 2

Problem 2.45 Find the electric field at a height z above the center of a square sheet (side a) carrying a uniform surface charge σ. Check your result for the limiting cases $a \to \infty$ and $z \gg a$.

$\left[\textit{Answer:} (\sigma/2\epsilon_0)\left\{(4/\pi)\tan^{-1}\sqrt{1+(a^2/2z^2)}-1\right\}\right]$

Problem 2.46 If the electric field in some region is given (in spherical coordinates) by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{k}{r}\left[3\,\hat{\mathbf{r}} + 2\sin\theta\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \sin\theta\cos\phi\,\hat{\boldsymbol{\phi}}\right],$$

for some constant k, what is the charge density? [*Answer:* $3k\epsilon_0(1+\cos 2\theta\sin\phi)/r^2$]

Problem 2.47 Find the net force that the southern hemisphere of a uniformly charged solid sphere exerts on the northern hemisphere. Express your answer in terms of the radius R and the total charge Q. [*Answer:* $(1/4\pi\epsilon_0)(3Q^2/16R^2)$]

Problem 2.48 An inverted hemispherical bowl of radius R carries a uniform surface charge density σ. Find the potential difference between the "north pole" and the center. [*Answer:* $(R\sigma/2\epsilon_0)(\sqrt{2}-1)$]

Problem 2.49 A sphere of radius R carries a charge density $\rho(r) = kr$ (where k is a constant). Find the energy of the configuration. Check your answer by calculating it in at least two different ways. [*Answer:* $\pi k^2 R^7/7\epsilon_0$]

Problem 2.50 The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r},$$

where A and λ are constants. Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(r)$, and the total charge Q. [*Answer:* $\rho = \epsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2 e^{-\lambda r}/r)$]

Problem 2.51 Find the potential on the rim of a uniformly charged disk (radius R, charge density σ). [*Hint:* First show that $V = k(\sigma R/\pi\epsilon_0)$, for some dimensionless number k, which you can express as an integral. Then evaluate k analytically, if you can, or by computer.]

! **Problem 2.52** Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$ (Fig. 2.54).

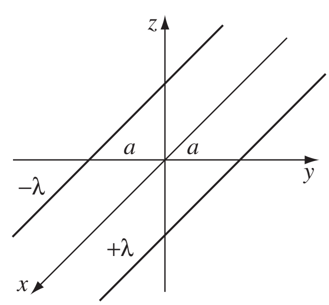

FIGURE 2.54

(a) Find the potential at any point (x, y, z), using the origin as your reference.

(b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential V_0.

! **Problem 2.53** In a vacuum diode, electrons are “boiled” off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

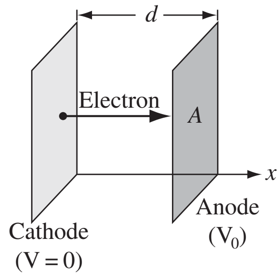

FIGURE 2.55

(a) Write Poisson’s equation for the region between the plates.

(b) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

(c) In the steady state, I is independent of x. What, then, is the relation between ρ and v?

(d) Use these three results to obtain a differential equation for V, by eliminating ρ and v.

(e) Solve this equation for V as a function of x, V_0, and d. Plot $V(x)$, and compare it to the potential *without* space-charge. Also, find ρ and v as functions of x.

(f) Show that

$$I = K V_0^{3/2}, \tag{2.56}$$

and find the constant K. (Equation 2.56 is called the **Child-Langmuir law**. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is *nonlinear*—it does not obey Ohm’s law.)

! **Problem 2.54** Imagine that new and extraordinarily precise measurements have revealed an error in Coulomb's law. The *actual* force of interaction between two point charges is found to be

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{\mathscr{r}^2}\left(1 + \frac{\mathscr{r}}{\lambda}\right) e^{-(\mathscr{r}/\lambda)}\,\hat{\boldsymbol{\mathscr{r}}},$$

where λ is a new constant of nature (it has dimensions of length, obviously, and is a huge number—say half the radius of the known universe—so that the correction is small, which is why no one ever noticed the discrepancy before). You are charged with the task of reformulating electrostatics to accommodate the new discovery. Assume the principle of superposition still holds.

(a) What is the electric field of a charge distribution ρ (replacing Eq. 2.8)?

(b) Does this electric field admit a scalar potential? Explain briefly how you reached your conclusion. (No formal proof necessary—just a persuasive argument.)

(c) Find the potential of a point charge q—the analog to Eq. 2.26. (If your answer to (b) was "no," better go back and change it!) Use ∞ as your reference point.

(d) For a point charge q at the origin, show that

$$\oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int_{\mathcal{V}} V\, d\tau = \frac{1}{\epsilon_0} q,$$

where $\mathcal{S}$ is the surface, $\mathcal{V}$ the volume, of any sphere centered at q.

(e) Show that this result generalizes:

$$\oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int_{\mathcal{V}} V\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}},$$

for *any* charge distribution. (This is the next best thing to Gauss's Law, in the new "electrostatics.")

(f) Draw the triangle diagram (like Fig. 2.35) for this world, putting in all the appropriate formulas. (Think of Poisson's equation as the formula for ρ in terms of V, and Gauss's law (differential form) as an equation for ρ in terms of $\mathbf{E}$.)

(g) Show that *some* of the charge on a conductor distributes itself (uniformly!) over the volume, with the remainder on the surface. [*Hint:* **E** is still zero, inside a conductor.]

Problem 2.55 Suppose an electric field $\mathbf{E}(x, y, z)$ has the form

$$E_x = ax, \qquad E_y = 0, \qquad E_z = 0$$

where a is a constant. What is the charge density? How do you account for the fact that the field points in a particular direction, when the charge density is uniform? [This is a more subtle problem than it looks, and worthy of careful thought.]

Problem 2.56 All of electrostatics follows from the $1/r^2$ character of Coulomb's law, together with the principle of superposition. An analogous theory can therefore be constructed for Newton's law of universal gravitation. What is the gravitational energy of a sphere, of mass M and radius R, assuming the density is uniform? Use your result to estimate the gravitational energy of the sun (look up the relevant numbers). Note that the energy is *negative*—masses *attract*, whereas (like) electric charges *repel*. As the matter "falls in," to create the sun, its energy is converted into other forms (typically thermal), and it is subsequently released in the form of radiation. The sun radiates at a rate of 3.86×10^{26} W; if all this came from gravitational energy, how long would the sun last? [The sun is in fact much older than that, so evidently this is *not* the source of its power.[14]]

! **Problem 2.57** We know that the charge on a conductor goes to the surface, but just how it distributes itself there is not easy to determine. One famous example in which the surface charge density can be calculated explicitly is the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

In this case[15]

$$\sigma = \frac{Q}{4\pi abc}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2}, \tag{2.57}$$

where Q is the total charge. By choosing appropriate values for a, b, and c, obtain (from Eq. 2.57): (a) the net (both sides) surface charge density $\sigma(r)$ on a circular disk of radius R; (b) the net surface charge density $\sigma(x)$ on an infinite conducting "ribbon" in the xy plane, which straddles the y axis from $x = -a$ to $x = a$ (let Λ be the total charge per unit length of ribbon); (c) the net charge per unit length $\lambda(x)$ on a conducting "needle," running from $x = -a$ to $x = a$. In each case, sketch the graph of your result.

Problem 2.58

(a) Consider an equilateral triangle, inscribed in a circle of radius a, with a point charge q at each vertex. The electric field is zero (obviously) at the center, but (surprisingly) there are three *other* points inside the triangle where the field is zero. Where are they? [*Answer:* $r = 0.285\,a$—you'll probably need a computer to get it.]

(b) For a regular n-sided polygon there are n points (in addition to the center) where the field is zero.[16] Find their distance from the center for $n = 4$ and $n = 5$. What do you suppose happens as $n \to \infty$?

[14] Lord Kelvin used this argument to counter Darwin's theory of evolution, which called for a much older Earth. Of course, we now know that the source of the Sun's energy is nuclear fusion, not gravity.
[15] For the derivation (which is a real *tour de force*), see W. R. Smythe, *Static and Dynamic Electricity,* 3rd ed. (New York: Hemisphere, 1989), Sect. 5.02.
[16] S. D. Baker, *Am. J. Phys.* **52**, 165 (1984); D. Kiang and D. A. Tindall, *Am. J. Phys.* **53**, 593 (1985).

Problem 2.59 Prove or disprove (with a counterexample) the following

> **Theorem:** Suppose a conductor carrying a net charge Q, when placed in an external electric field $\mathbf{E}_e$, experiences a force $\mathbf{F}$; if the external field is now reversed ($\mathbf{E}_e \rightarrow -\mathbf{E}_e$), the force also reverses ($\mathbf{F} \rightarrow -\mathbf{F}$).

What if we stipulate that the external field is *uniform*?

Problem 2.60 A point charge q is at the center of an uncharged spherical conducting shell, of inner radius a and outer radius b. *Question:* How much work would it take to move the charge out to infinity (through a tiny hole drilled in the shell)? [*Answer:* $(q^2/8\pi\epsilon_0)(1/a - 1/b)$.]

Problem 2.61 What is the minimum-energy configuration for a system of N equal point charges placed on or inside a circle of radius R?[17] Because the charge on a conductor goes to the surface, you might think the N charges would arrange themselves (uniformly) around the circumference. Show (to the contrary) that for $N = 12$ it is better to place 11 on the circumference and one at the center. How about for $N = 11$ (is the energy lower if you put all 11 around the circumference, or if you put 10 on the circumference and one at the center)? [*Hint:* Do it numerically—you'll need at least 4 significant digits. Express all energies as multiples of $q^2/4\pi\epsilon_0 R$]

[17] M. G. Calkin, D. Kiang, and D. A. Tindall, *Am. H. Phys.* **55**, 157 (1987).

CHAPTER

3 Potentials

3.1 ■ LAPLACE'S EQUATION

3.1.1 ■ Introduction

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau'. \tag{3.1}$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the *potential*, V, which is given by the somewhat more tractable Eq. 2.29:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\mathbf{r}')\, d\tau'. \tag{3.2}$$

Still, even *this* integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance; since charge is free to move around, the only thing we control directly is the *total* charge (or perhaps the potential) of each conductor.

In such cases, it is fruitful to recast the problem in differential form, using Poisson's equation (2.24),

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \tag{3.3}$$

which, together with appropriate boundary conditions, is equivalent to Eq. 3.2. Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. (If $\rho = 0$ *everywhere,* of course, then $V = 0$, and there is nothing further to say—that's not what I mean. There may be plenty of charge *elsewhere,* but we're confining our attention to places where there is no charge.) In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0, \tag{3.4}$$